Math Journal

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π day

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Specials

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Curious Facts.

BY *Isaac Li*

π, along with e (Euler's number) are two of the most well-known irrational and *transcendental* numbers!

Irrational numbers are those that cannot be written as the quotient of two *integers*. This means that there are no integers a, b such that

$$\pi = \frac{a}{b}.$$

Transcendental numbers are those that cannot be expressed as a root of any (nonzero) polynomial with rational coefficients. Note that a number can be irrational but not transcendental; consider the number $\sqrt{2}$. It is obviously irrational, but it's also a root to the polynomial

$$x^2 - 2 = 0.$$

Thus $\sqrt{2}$ is not transcendental.

At the time of writing it is not known whether

$$\pi^{\pi^{\pi^{\pi}}}$$

is an integer. The reason is that the constant is so big that even modern computers can't compute its value. Even if we could, it'd still be hard to prove its irrationality or transcendence.

There is also another class of troubling numbers, namely most sums, products, powers, etc. of π and e, such as:

$$\pi + e, \quad \pi - e, \quad \pi \cdot e, \quad \pi^{\pi}, \quad e^{\pi}$$

These are not known to be rational, algebraic, irrational or transcendental, which really shows that we are still in the stone age of mathematics.

There are many infinite sums and products that can be used to compute the value of π. Of the most famous we have

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\frac{\pi}{2} = \left(\frac{2}{1} \cdot \frac{2}{3}\right) \cdot \left(\frac{4}{3} \cdot \frac{4}{5}\right) \cdot \left(\frac{6}{5} \cdot \frac{6}{7}\right) \cdot \dots$$

$$\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \dots$$

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \dots \tag{4}$$

These formulae demonstrate the unexpected and beautiful connections between different areas of mathematics. As such we have made a separate entry explaining (4).

Many online sources claim that it is possible to find any sequence of digits in the expansion of π — indeed we can find for example (in base 27):

“PIE” at position 21913

“SJC” at position 28010

“MATH” at position 911592

“JOSEPH” at position 117295852

However, this does not prove anything, unless we can check the infinitely many combinations of letters (which we obviously can't). It may come as a bit of a surprise that mathematicians have yet to prove that π is *normal* — i.e. all possible strings of any finite length can and will occur randomly (this is crudely oversimplified). However it is widely believed to be true, we just do not have a proof yet.

For more fun with the digits of π, check out the next entry.

(The reader is kindly asked to take this purely in a humorous vein.) According to the BIBLE, $\pi = 3$ — and I cite KINGS 7:23 :

> 23 ¶ *And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about.*

Clearly, $30/10 = 3 = \pi$! Of course, this doesn't take into account the physical thickness of objects, which is probably why the value obtained here differs from the actual value of 3.1415 . . .

Non π-related

There are different orders of infinities! Cantor proved that the cardinality of the set of real numbers $\mathbb{R}$ is *uncountably* infinite. This differs from, and is strictly greater than, the cardinalities of sets such as the natural numbers $\mathbb{N}$, integers $\mathbb{Z}$ and rational numbers $\mathbb{Q}$, all of which are *countably* infinite.

Thanks to the Gödel's *Incompleteness Theorems*, any axiomatic system (A formal axiomatic system is a set of axioms, i.e. absolute assumed truths, from which all theorems are to be derived from.) that is capable of modelling arithmetic cannot prove its own consistency! Of course, you can carry out one in a *stronger* system, however the consistency of said system itself would be another problem, and so on *ad infinitum*. Thus we can never truly know whether the foundations of mathematics is consistent.

A statement P is either true or false . . . or is it? In classical logic this is known as the *law of the excluded middle*. However, certain statements have been found to be *independent* (from a set of axioms) — that means they can neither be proven nor disproved. Perhaps the most (in)famous example is CH (Continuum Hypothesis), which Cohen proved to be independent from the standard set theoretic axiom system ZFC using a technique called *forcing*.

Iterating the Digits of π.

BY *Raphael Li*

It is well known that the digits of π starts with

$$3.1415926\ldots$$

and goes on and on forever. Each digit, ignoring the ones that come before the decimal point, can be labelled with an index like so:

Index	1	2	3	4	5	6	7	...
Digit	1	4	1	5	9	2	6	...

We can then define a function $f(s)$ as follows:

- The input s must be a string of digits[1];
- If s appears in the first one million digits[2] of π, $f(s)$ is defined as the index of its first occurrence, converted to a string;
- If s does not appear in the first one million digits of π, $f(s) = s$.

For instance, $f(\texttt{59}) = \texttt{4}$ because the string of digits `59` first appears at index 4. On the other hand, we let $f(\texttt{123456}) = \texttt{123456}$ as the string `123456` doesn't appear[3] in the first one million digits of π. Feel free to play around with this function at http://pi.fathom.info.

Now that we have a function, what happens when we iterate it? In general, iterating a function $f(x)$ involves the following steps:

[1] This implies that leading zeroes must not be ignored. For example, $f(\texttt{00123}) \neq f(\texttt{123})$.

[2] Here, one million is completely arbitrary.

[3] For the curious, the said string doesn't appear until the $\texttt{2458885}^{\text{th}}$ position.

1. Choose an initial value a_0.
2. Compute $a_1 = f(a_0)$.
3. Compute $a_2 = f(a_1)$.

   And so on.

To illustrate, here's what happens when we let $a_0 = \texttt{03142021}$ and $a_0 = \texttt{211}$ respectively:

$$\begin{aligned} f(\texttt{03142021}) &= \texttt{589213} \\ f(\texttt{589213}) &= \texttt{663943} \\ f(\texttt{663943}) &= \texttt{781568} \\ f(\texttt{781568}) &= \texttt{182162} \\ f(\texttt{182162}) &= \texttt{182162} \end{aligned} \qquad \begin{aligned} f(\texttt{211}) &= \texttt{93} \\ f(\texttt{93}) &= \texttt{14} \\ f(\texttt{14}) &= \texttt{1} \\ f(\texttt{1}) &= \texttt{1} \end{aligned}$$

As we can see, the sequence on the left[4] reaches a loop[5] after 5 iterations. We say that the string 03142021 has a *persistence* of 5. We will denote this as $P(\texttt{03142021}) = 5$. Similarly, $P(\texttt{211}) = 4$. It is worth noting that the sequence on the right loops not because 1 doesn't appear in the first one million digits of π, but because of the coincidence that 1 first appears at index 1. These are called *self-locating strings*[6].

The fact that different starting values have different persistence raises the question: Which string has the greatest persistence? Using code (included at the end of this article), I have checked through every string that represents a positive integer less than one million[7] in about half an hour.

[4] It just so happens that the date of this year's Pi Day (03142021) appears in the first one million digits of π, which is rare. The next time this happens will be over a century later – Pi Day 2177 (03142177). Isn't this worth a celebration?

[5] Loops may not necessarily be of length 1. For example, $s_a \to s_b \to s_c \to s_a$ also counts as a loop.

[6] Known self-locating strings in π are listed in sequence A057680 in the OEIS. Learn more about them in this Numberphile video: https://youtu.be/W20aT14t8Pw

[7] In other words, the program looks through strings 1 through 999999 and finds the string with maximum persistence. Note that strings with leading zeroes, such as 000314, are not checked. If you're into Python, you can try editing the code at the end of this article to include those strings as well.

But before we reveal the string with the greatest persistence, let's take a look at a few interesting facts I found out during the testing of the program.

$$P(\texttt{77}) = 7 \qquad P(\texttt{1357}) = 9$$
$$P(\texttt{1818}) = 18 \qquad P(\texttt{23456}) = 7$$

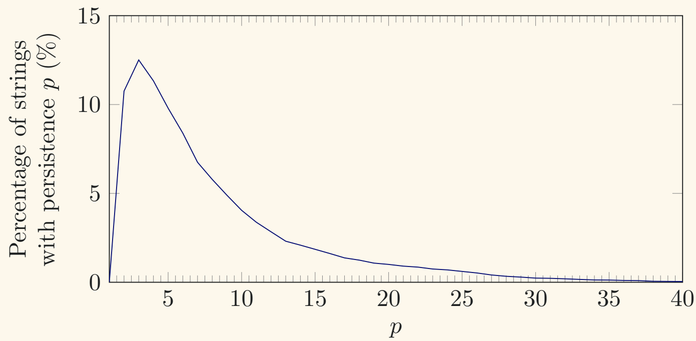
In addition to the above, there are various solutions to the equation $P(s) = s$, including, but not limited to, 00003, 08, 13, 0022 and 23.

But still, one question remains unanswered: which string has the maximum persistence? Here comes the leaderboard:

Place	String	Persistence
1	329347	61
2	238480	60
3	158839	59

61 iterations. That's a lot. If you have the time, try finding one with even more iterations. As with other similar pieces of math, the question most people ask is: How is this useful? To which I say, it's recreational mathematics – it's meant to be useless. But useless math is the best and the most beautiful kind of math, isn't it?

But I digress. As an exercise to the reader, take a look at the following piece of data collected whilst running the code – what happens when we iterate $P(s)$?



The code described in this article is included below. A text file (named `pi.txt`) containing 1 million digits of π must be included[8] in the project folder in order for it to work.

```python
from time import time

piString = open("pi.txt", "r").read()[2:]


def get_index(string):
    global piString
    return piString.index(string)+1


def seq_length(n, printing=False):
    seq = [str(n)]

    if printing:
        print("\n"+str(seq[0]))

    while True:
        try:
            seq.append(str(get_index(seq[-1])))
            if printing:
                print(seq[-1])
            if seq.count(seq[-1]) > 1:
                if printing:
                    print("Loop!")
                break
        except ValueError:
            if printing:
                print("End of sequence")
            break

    return len(seq)


if __name__ == '__main__':
    record = 0
    low = int(input("Enter lower bound of search range (inclusive): "))
    high = int(input("Enter upper bound of search range (exclusive): "))

    start_time = time()
    minutes_passed = 0

    for curr in range(low, high):
        length = seq_length(curr)
        if length >= record:
            print(f"New record: a0 = {curr} --> loops after {length} iterations")
            record = length
        if (time() - start_time)/60 > minutes_passed + 1:
            minutes_passed += 1
            print(f"{minutes_passed} minutes passed; Progress: {round(100*(curr - low)/(
  high - low))}%")
    print("\nDone")
```

[8]Such a file can be downloaded at http://pi2e.ch/blog/wp-content/uploads/2017/03/pi_dec_1m.txt.

Viète's Formula.

BY *Isaac Li* and *Raphael Li*

Note: Below is an informal explanation of Viète's Formula and is by no means a rigorous proof; for readers that seek one, textbooks are a much better source.

LEONHARD EULER was perhaps one of the most influential mathematicians. Amongst his many discoveries and developments, Euler is credited for popularizing the Greek letter π to denominate the Archimedes' constant, the *pi* we all know and love today; as such, we shall look at one of his derivations for an infinite series involving π, *Viète's Formula*[9].

Viète's formula is as follows:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

(This is, in fact, the first ever infinite product in the history of mathematics.)
In this entry we present two proofs: a geometric argument; and an algebraic derivation by Euler.

The Geometric View

This argument will be separated into two parts:

1. Proving a variation of the formula geometrically; and

2. Showing its equivalence to the original.

[9] Some readers may know of another (perhaps more conventional in the context of introducing Euler's many wonderful discoveries) infinite series, namely Euler's solution to the *Basel Problem*. However, I choose not to introduce it here for two reasons: First, I wanted to keep this article as short as possible, and that means favouring topics less demanding on prerequisites while still providing an adequate explanation; The second is that there are already ample resources on the Internet for the Basel Problem, so I shall leave it to the reader (One I particularly recommend is 3B1B's YouTube video, `https://youtu.be/d-o3eB9sfls`).

Proving a Variation of the Formula

In this section we will show that

$$\lim_{n\to\infty} 2^n \cdot \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}}}_{n \text{ square roots}} = \pi. \qquad (*)$$

To do so, imagine a regular polygon with 2^{n+1} sides[10] being inscribed in a unit circle[11]. As n grows larger and larger, the perimeter of the polygon will get closer and closer to the circumference of the circle. To calculate the perimeter of the polygon, we can simply multiply its side length by the number of sides – the latter is just 2^{n+1}, but the former is a little bit more difficult to figure out.

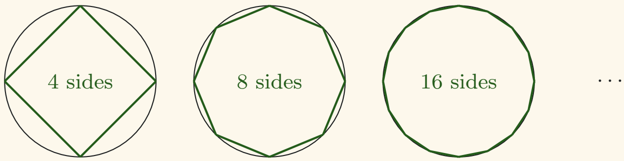
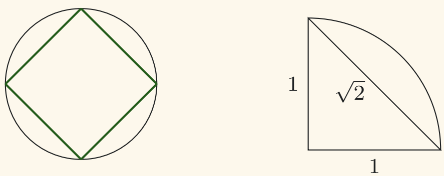


Figure 1: The number of sides of the polygon doubles every iteration.

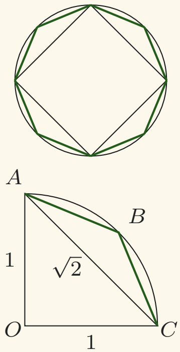
Let us start with *the first iteration*, where a square is inscribed in a circle (see left figure). If we focus on the upper-right quadrant of the circle (see right figure), we can see that, since the radius is 1, by the Pythagorean theorem, the side length of the square s_1 is $\sqrt{2}$.

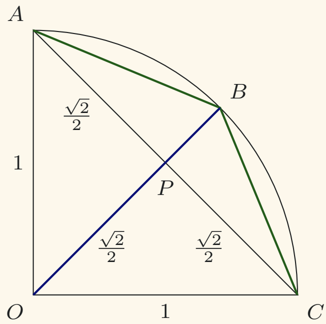


[10] The reason why I've written this as 2^{n+1} instead of just 2^n is that in the n-th iteration, the regular polygon will have a total of 2^{n+1} sides, not 2^n.

[11] A circle with radius 1.

In *the second iteration*, we will be dealing with a regular octagon instead of a square.





To find its side length s_2, we will connect OB, which intersects AC at P (coloured in blue). It follows that:

$$
\begin{aligned}
AP = PC &\Longrightarrow PC = \frac{s_1}{2} = \frac{\sqrt{2}}{2} \\
&\Longrightarrow OP = \sqrt{OC^2 - PC^2} = \frac{\sqrt{2}}{2} && \text{(Pyth. thm.)} \\
&\Longrightarrow PB = 1 - OP = 1 - \frac{\sqrt{2}}{2} \\
&\Longrightarrow BC = \sqrt{PC^2 + PB^2} = \sqrt{2 - \sqrt{2}} && \text{(Pyth. thm.)} \\
&\Longrightarrow s_2 = \sqrt{2 - \sqrt{2}}
\end{aligned}
$$

It can be seen from the first line that finding the value of s_2 requires finding that of s_1 first. In fact, if we let s_n denote the side length of the polygon in the n-th iteration, we can express the value of s_n as a recursive formula:

$$
s_n = \sqrt{\left(1 - \sqrt{1^2 - \left(\frac{s_{n-1}}{2}\right)^2}\right)^2 + \left(\frac{s_{n-1}}{2}\right)^2}
$$

where the blue part and the red part are analogous to the lengths of OP and PC respectively. Simplifying the expression gives:

$$s_n = \sqrt{2 - \sqrt{4 - s_{n-1}^2}}\,.$$

which produces the following values:

$$\begin{aligned} s_1 &= \sqrt{2} \\ s_2 &= \sqrt{2 - \sqrt{2}} \\ s_3 &= \sqrt{2 - \sqrt{2 + \sqrt{2}}} \\ s_4 &= \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \\ &\vdots \\ s_n &= \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{2}}}}}}_{n \text{ square roots}} \end{aligned}$$

Therefore,

$$\begin{aligned} \pi = \frac{\text{Circumference}}{2\,(\text{Radius})} &= \frac{\text{Circumference}}{2} && \text{(by definition)} \\ &= \lim_{n\to\infty} \frac{s_n \cdot 2^{n+1}}{2} \\ &= \lim_{n\to\infty} s_n \cdot 2^n \\ &= \lim_{n\to\infty} 2^n \cdot \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{2}}}}}}_{n \text{ square roots}}. \end{aligned}$$

Showing the Variation's Equivalence to the Original

In this part we will show that equation $(*)$ is actually equivalent to Viète's formula. Consider the following equation:

$$\left(\frac{a_1}{a_2}\right)\left(\frac{a_2}{a_3}\right)\left(\frac{a_3}{a_4}\right)\cdots\left(\frac{a_{n-1}}{a_n}\right)=\frac{a_1}{a_n} \tag{1}$$

This technique of cancelling out terms is called *telescoping*[12]. Building upon this, let us start our (somewhat non-rigorous) proof.

Proof. Let $P(n)$ denote the perimeter of a regular 2^{n+1}-gon inscribed within a unit circle. In other words, $P(n) = s_n \cdot 2^{n+1}$. We know that

$$\lim_{n\to\infty} P(n) = \text{Circumference} = 2\pi, \tag{2}$$

as illustrated by figure 1. Thus, we have

$$\begin{aligned}
\frac{2}{\pi} &= \frac{4}{2\pi} \\
&= \lim_{n\to\infty} \frac{4}{P(n)} && \text{(by (2))} \\
&= \lim_{n\to\infty} \left(\frac{4}{P(1)}\right)\left(\frac{P(1)}{P(2)}\right)\left(\frac{P(2)}{P(3)}\right)\cdots\left(\frac{P(n-1)}{P(n)}\right) && \text{(by (1))} \\
&= \frac{4}{4\sqrt{2}} \cdot \frac{4\sqrt{2}}{8\sqrt{2-\sqrt{2}}} \cdot \frac{8\sqrt{2-\sqrt{2}}}{16\sqrt{2-\sqrt{2+\sqrt{2}}}} \cdots \\
&= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \ldots
\end{aligned}$$

□

[12] This technique is named telescoping (formally known as the *method of differences*) because the terms in the sum or product collapse (cancel out) like a telescope does, leaving the first and last terms behind. Such a sum or product is called a *telescoping series*.

An Algebraic Derivation

To start off, we have to introduce a concept called "mathematical induction" — the idea's very simple really: We have a statement $Q(n)$ on the natural numbers (namely 0, 1, 2, 3, $\cdots$), and we wish to prove it for *all* natural numbers.
Such a proof is (typically) split into two steps:

1. The base case — we prove the statement for 1. Simple enough;
2. The inductive step — we show that $Q(n)$ implies $Q(n+1)$.

It is helpful to think of it as the sequential effect of falling dominoes: Push the first one ($Q(0)$), and the rest ($Q(1)$, $Q(2)$, $Q(3)$, $\cdots$) all come falling down.

Trigonometric Trickery[13]

We begin with the double-angle formula:

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}.$$

What happens when we iterate it?

$$\begin{aligned}\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2}\\ &= 2^2\sin\frac{x}{2^2}\cos\frac{x}{2^2}\cos\frac{x}{2},\end{aligned}$$

and once more:

$$= 2^3\sin\frac{x}{2^3}\cos\frac{x}{2^3}\cos\frac{x}{2^2}\cos\frac{x}{2}.$$

At this point we may notice a pattern: at the n-th iteration the formula will be:

$$\begin{aligned}&2^n\sin\frac{x}{2^n}\overbrace{\cos\frac{x}{2^n}\cos\frac{x}{2^{n-1}}\cdots\cos\frac{x}{2^2}\cos\frac{x}{2^1}}^{n\text{ times}}\\ &= 2^n\sin\frac{x}{2^n}\prod_{k=1}^{n}\cos\frac{x}{2^k}.\end{aligned}$$

Notice the subtle use of induction in the argument above — at the $(n+1)$-th iteration we make use of the step before and apply the formula again to derive following stages. The inductive step above may be formalized as follows:

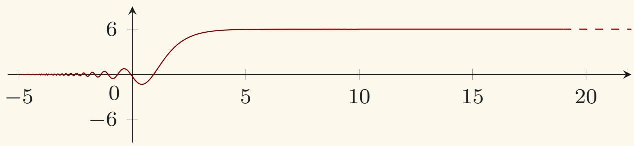
$$
\begin{aligned}
&2^n \sin \frac{x}{2^n} \prod_{k=1}^{n} \cos \frac{x}{2^k} \\
&= 2^n \cdot 2 \sin \frac{x/(2^n)}{2} \cos \frac{x/(2^n)}{2} \prod_{k=1}^{n} \cos \frac{x}{2^k} \\
&= 2^{n+1} \sin \frac{x}{2^n \cdot 2} \cos \frac{x}{2^n \cdot 2} \prod_{k=1}^{n} \cos \frac{x}{2^k} \\
&= 2^{n+1} \sin \frac{x}{2^{n+1}} \prod_{k=1}^{n+1} \cos \frac{x}{2^k}
\end{aligned}
\qquad \square
$$

And that was why I felt the need to introduce induction at the start of this section (particularly in an algebraic derivation).

Now, what happens when we take the *limit* as n goes to infinity? Intuitively, this just means taking the value of an expression as n gets larger and larger. Let's look at

$$\lim_{n\to\infty} 2^n \sin \frac{x}{2^n} = \lim_{n\to\infty} \frac{\sin(x/2^n)}{2^{-n}}.$$

The red parts get infinitely small as n approaches infinity. This is actually a variant of another famous limit $\lim_{n\to 0}(\sin x)/x$, which evaluates to x. Unfortunately due to the scarcity of space you will have to take my word for this one, but they return the same result. Below I plot a graph of $\lim_{n\to\infty}(\sin(x/2^n))/(2^{-n})$ with $x = 6$:



[13] If you're unfamiliar with trigonometry or limits, but are happy with an argument as it stands, independently of the reference, then feel free to move on.

Indeed, the right hand side approaches exactly 6 (I hope you are convinced now). Using this limit, we may write

$$\begin{aligned}
\sin x &= \lim_{n\to\infty} 2^n \sin\frac{x}{2^n} \prod_{k=1}^{n} \cos\frac{x}{2^k} \\
&= x \cdot \lim_{n\to\infty} \prod_{k=1}^{n} \cos\frac{x}{2^k} \\
&= x \prod_{k=1}^{\infty} \cos\frac{x}{2^k} \\
&= x \cdot \cos\frac{x}{2} \cos\frac{x}{2^2} \cos\frac{x}{2^3} \cdots .
\end{aligned}$$

Dividing both sides by x:

$$\frac{\sin x}{x} = \cos\frac{x}{2} \cos\frac{x}{2^2} \cos\frac{x}{2^3} \cdots . \qquad (*)$$

Now, recall the cosine half-angle formula:

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}.$$

Take $x = \pi/2$; then

$$\begin{aligned}
\cos\frac{\pi}{4} &= \sqrt{\frac{1+\cos(\pi/2)}{2}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}, \\
\cos\frac{\pi}{8} &= \sqrt{\frac{1+\cos(\pi/4)}{2}} = \sqrt{\frac{1+\sqrt{2}/2}{2}} \\
&= \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}, \\
&\vdots \\
\cos\frac{\pi}{2^n} &= \sqrt{\frac{1+\cos(\pi/2^{n-1})}{2}} = \frac{\overbrace{\sqrt{2+\sqrt{2+\sqrt{\cdots\sqrt{2+\sqrt{2}}}}}}^{(n-1)\text{ square roots}}}{2}.
\end{aligned}$$

The reader can quickly verify the general form by induction.

For the final step, we substitute the above into (∗), which becomes

$$\begin{aligned}\frac{\sin(\pi/2)}{\pi/2} &= \frac{2}{\pi}\\ &= \cos\frac{\pi/2}{2}\cos\frac{\pi/2}{2^2}\cos\frac{\pi/2}{2^3}\cdots\\ &= \cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}\cos\frac{\pi}{2^4}\cdots\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdot\cdots\end{aligned}$$

And we are done.

Happy π Day!