ST. JOSEPH'S COLLEGE **MATHEMATICS SOCIETY**

MATHS JOURNAL

A collection of expository mathematics articles

Discover the beauty of mathematics with us

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Foreword

It is a tradition of the Society to publish an expository journal dedicated to the communication of interesting mathematics annually. This year, we have assembled contributions from four executive commitee members, as well as our former president Toby Lam. We hope the articles contained herein will educate as well as entertain you, regardless of your background in mathematics.

Isaac Li
President of the Mathematics Society

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The 5 Platonic Solids

BY MARCO CHIU

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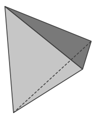
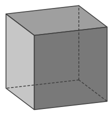


Cube



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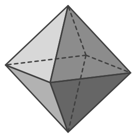
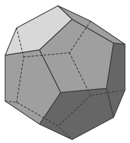


Dodecahedron



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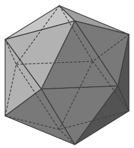


What are the Platonic Solids?

Platonic solids are polyhedra whose faces, edges, and vertices are all the same as each other.

This means that every face must be a regular polygon, having the same number of sides as every other face, and at every vertex, the same number of edges and faces meet. For example, in the dodecahedron, all the faces are regular pentagons. At all the vertices, 3 pentagons and 3 edges meet.

Since there is only one type of face on a Platonic solid (same for the edges and vertices), they are very symmetric. Some of them are even used as dice.

Proof That There are Only 5 Platonic Solids

We attempt to find some properties a Platonic solid must have, and thus prove that only 5 Platonic solids exist.

By definition, for every Platonic solid:

1. Every face has the same number, s, of sides.

2. Every vertex touches the same number, n, of edges.

Let F, E, and V be, respectively, the number of faces on the Platonic solid, the number of edges it has, and the number of vertices it has. By Euler's polyhedral formula , we get $F - E + V = 2$ for any convex polyhedron.

Not to be confused with Euler's theorem, nor the Local Euler characteristic formula

Notice that if you try to count the number of edges of the polyhedron using $s \cdot F$, each edge is counted exactly twice, since every edge is an edge of two faces, so the sum of (the number of edges on each face) is 2E. Therefore, $sF = 2E$. Similarly, every edge touches two vertices, so the sum of the number of edges touching each vertex is $2E$. So we have $nV = 2E$. Using these, we can rewrite Euler's formula in terms of E, s and v:

$$\begin{aligned} F - E + V &= 2 \\ \frac{2E}{s} - E + \frac{2E}{n} &= 2 \\ \frac{2}{s} + \frac{2}{n} &= 1 + \frac{2}{E} \end{aligned}$$

Since we know $E > 0$, the RHS is greater than 1, we can rearrange to get $1/s + 1/n > 1/2$.

Notice that if $s \geqslant 6$, then $n < 3$, which is absurd. So, Platonic solids can't be made of hexagons. Also note that $s \geqslant 3$. Since the s and n are interchangeable, we also have $3 \leqslant n \leqslant 5$.

What is left now is to check all the possible combinations of s and n. Indeed, we find that the pairs that satisfy the relation above are as follows:

$$\begin{aligned} (s, n) = &(3,3) \text{ Tetrahedron}, \quad (4,3) \text{ Cube}, \\ &(3,4) \text{ Octahedron}, \quad (5,3) \text{ Dodecahedron}, \\ &(3,5) \text{ Icosahedron} \end{aligned}$$

These 5 pairs correspond exactly to the 5 Platonic solids.

Why is This Proof Important?

This line of reasoning can be extended to other types of graphs. For example, this method of proof can be used to prove that on a toroidal graph which is made up of only pentagons, hexagons and heptagons, and whose vertices touch exactly 3 edges, there must be the same number of pentagons as heptagons. For more, see `http://origametry.net/combgeom/tori/torusnotes.html`.

Mathematics and the Arts

BY ANTHONY LAI

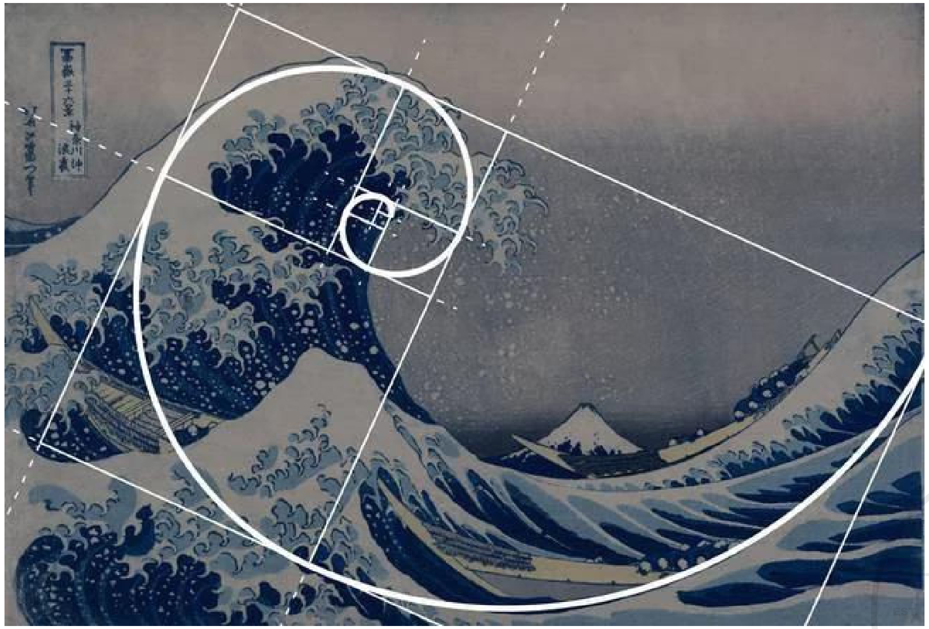
Mathematics and arts are widely regarded as two opposite subjects, with the former focusing on logical reasoning and critical thinking, and the latter putting an emphasis on the expression of emotions, idea or a world view. As different as these two areas of study seem, the aesthetic beauty of arts has a close relationship with mathematics.

The golden ratio is a beautiful constant in mathematics defined as $\frac{1+\sqrt{5}}{2}$.[1] It can be derived from the Fibonacci sequence. The sequence starts with 1 and 1, and each proceeding term is obtained by summing up the previous two terms. The ratio between two successive terms will converge to the golden ratio.[2] The golden ratio is often associated with the golden spiral, which is constructed using squares with the lengths of Fibonacci numbers that are connected together in a spiral. A quarter circle is then drawn in each square to form this golden spiral. An example of an artwork which utilises the beauty of this ratio, is *The Great Wave off Hanagawa* by Katsushika Hokusai, who is famous for his geometrically precise works, as the ocean waves in the painting form a spiral roughly composed of quarter circles of different sizes, similar to the golden spiral.[3]

[1] Dunlap, Richard A. *The golden ratio and Fibonacci numbers*. World Scientific, 1997.

[2] Schneider, Robert, *Fibonacci Numbers and the Golden Ratio*, ArXiv.org, 2016

[3] Powera, Seamus A., and Anthony G. Shannonb. *Natural Mathematics, the Fibonacci Numbers and Aesthetics in Art*.



The usage of different sizes of wave spirals allows for the golden ratio to manifest itself throughout the painting between different sizes of quarter circles and improves the overall aesthetic of the painting. The golden ratio is widely used in the art world with examples of its applications found in Dali's Sacrament of the *Last Supper*, and also in *Modulor*, designed by architect Le Corbusier.[4]

The golden ratio can also be found in many portraits because of its correlation to natural beauty and its ubiquity in nature, for instance, snail shells and nautilus shells follow the golden spiral, as well as the shape of the ear of a human body.[5] In the famous portrait, the *Mona Lisa* by Da Vinci, the golden spiral can be found on the left side of the woman's face, and simultaneously, the anticlockwise spiral beautifully frames her face. The spiral winds from her nose,

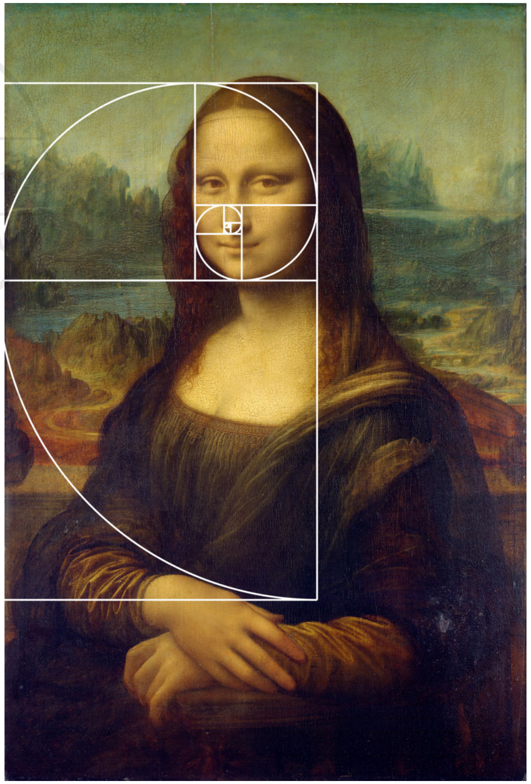
[4] Livio, Mario. *The golden ratio and aesthetics*. Plus Magazine 22 (2002).

[5] Persaud, Dharam, and James P. O'Leary. *Fibonacci series, golden proportions, and the human biology*. (2015).

wraps around her chin, and with one standard line, connects her elbow to her thumb.[6] The golden ratio creates an aesthetically pleasing appearance due to its prominence in nature, and its role in establishing natural beauty.

The golden ratio is an invaluable asset to art and beauty, but almost an oddity to the art world as it is mainly constructed by taking the ratio of two integers, or solving a simple quadratic equation, who would've thought that the basic numeric and natural patterns of our world, give an aesthetically pleasing and beautiful ratio which creates the basis of mankind's artistic creations.

[6] Foundation, Mona Lisa, *Leonardo and Mathematics*, The Mona Lisa Foundation, 2018



HKDSE Physics and M2

BY TOBY LAM

This article was originally published on Toby Lam's blog, which hosts a variety of content on topics ranging from mathematics, technology, to life at Oxford and career advice. For more, visit `tobylam.xyz`*.*

A lot of the formulae[7] given to you in HKDSE Physics, as it turns out, can be derived from the calculus taught in M2. In this series of posts we're going to go through deriving some of them. For a more detailed treatise on this topic, I would highly recommend checking out the dynamics lecture notes[8], which is a course for first year mathematics at Oxford.

[7] https://www.hkeaa.edu.hk/DocLibrary/HKDSE/Subject_Information/phy/Phy-Formulae-e.pdf

[8] https://courses.maths.ox.ac.uk/pluginfile.php/3628/mod_resource/content//DynamicsLectureNotes2022_updated.pdf

We would look at rectilinear motion in part I, projectile/circular motion in part II and waves in part III.

Part I: Rectilinear Motion

Rectilinear motion is one-dimensional motion along a straight line. Due to it only having one dimension, all properties about the system could be represented by one variable only. We wouldn't need to deal with coordinates.

Consider some point particle with constant mass m. As we've seen in M2, we can respectively let displacement, ve-

locity and acceleration be functions of time

$$\begin{aligned} \text{Displacement } &= r(t) \\ \text{Velocity } &= v(t) = \frac{dr}{dt} \\ \text{Acceleration } &= a(t) = \frac{d^2r}{dt^2}. \end{aligned}$$

Under this language, we can reframe Newton's First law as

$$\text{Momentum } = p(t) = mv(t) = m\frac{dr}{dt}$$

and Newton's second law as

$$\text{Force } = F(t) = \frac{dp}{dt} = m\frac{dv}{dt} = ma.$$

Introducing Assumptions

To get any further, we need to introduce some assumptions in DSE physics. In rectilinear motion we assume that

1. Force is constant (e.g. gravitational force)

This means that acceleration is constant! We would now write $a(t)$ as a as it's just a constant. This is crucial as it means that

$$\begin{aligned} \frac{d^2r}{dt^2} &= a \\ \frac{dr}{dt} &= at + C_1 \\ r(t) &= \frac{1}{2}at^2 + C_1t + C_2 \end{aligned}$$

by repeated indefinite integration for some constants C_1, C_2. Naturally, we ask what those constant are. We can see that

$$\begin{aligned} v(0) &= a \cdot 0 + C_1 = C_1 \\ r(0) &= \frac{1}{2}a \cdot 0 + C_1 \cdot 0 + C_2 = C_2 \end{aligned}$$

So C_1 is the velocity at $t = 0$. C_2 is the displacement at $t = 0$, which is generally taken to be 0.

Finally putting it all together we have

$$\begin{aligned} v(t) &= at + v(0) \\ r(t) &= \frac{1}{2}at^2 + v(0)t + r(0). \end{aligned}$$

Does this look familiar?

Conservation of Energy

To see why energy is conserved, we must first define the kinetic energy of a point particle at time t to be

$$T(t) = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2$$

and the potential energy for a point particle with displacement r (under constant force) to be

$$V(r) = -mar.$$

From DSE physics, we know that energy is conserved. I.e. $T + V$ is kept constant. However this is rather unobvious. Note how kinetic energy is with respect to time, but potential energy is with respect to displacement. In general, why would something with respect to time be related to something with respect to displacement?

It turns out that for energy to be conserved, the force needs to be conservative. In the one dimensional case, this means that there must exist a potential energy function $V(r)$ such that $F(r) = -\frac{d}{dr}V(r)$. This also means that the force is dependent on displacement only: If you are at the same displacement at different times, the force experienced is still the same.

For the case of DSE physics, as the acceleration/force is kept constant we could have $V(r) = -mar$, so the force is conservative. Note how we can add any constant to $V(r)$ and it would still be a valid potential function. Refer to the dynamics lecture notes for a more general analysis on conservative forces.

Now how do we show conservation of energy for this specific case? There's two ways of doing it. Either we expand all the terms as follows

$$\begin{aligned} (T + V) &= \left[\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 - mar\right] \\ &= m\left[\frac{1}{2}\left(at + v(0)\right)^2 - a\left(\frac{1}{2}at^2 + v(0)t + r(0)\right)\right] \\ &= m\left[\frac{1}{2}a^2t^2 + v(0)at + \frac{1}{2}v(0)^2 - \frac{1}{2}a^2t^2 - av(0)t - ar(0)\right] \\ &= \frac{1}{2}mv(0)^2 - mar(0) \end{aligned}$$

Or we can do it more abstractly by considering the derivative of $T + V$

$$\begin{aligned} \frac{d}{dt}(T + V) &= \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + V(r)\right] \\ &= \frac{1}{2}m \cdot 2\frac{d^2r}{dt^2} \cdot \frac{dr}{dt} + \frac{dV}{dr}\frac{dr}{dt} \\ &= m\frac{d^2r}{dt^2} \cdot \frac{dr}{dt} - m\frac{d^2r}{dt^2} \cdot \frac{dr}{dt} \\ &= 0 \end{aligned}$$

product and chain rule

$-m\frac{d^2r}{dt^2} = -F(r) = \frac{dV}{dr}$

So $T + V$ is constant.

In particular, this means that

$$\frac{1}{2}mv(t)^2 - mar(t) = \frac{1}{2}mv(0)^2 - mar(0).$$

Does this look familiar?

Part II: Projectile & Circular Motion

We would now look into projectile motion and uniform circular motion.

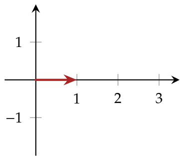
I would highly recommend checking out a video on vectors[9] before reading the post. Having a general idea of what vectors are would be extremely helpful.

[9] https://youtu.be/fNk_zzaMoSs

Motion on the 2D Plane

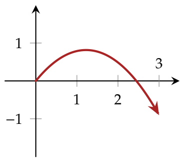
To study motion on the 2D plane, we need the idea of curves. The trajectory of a moving particle naturally forms a curve as time varies.

Mathematically, we model a curve as a function from $\mathbb{R}$, the real numbers, to $\mathbb{R}^2$, the cartesian plane. Here are some examples below.

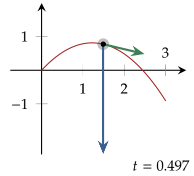


Straight Line. The function $r(t) = (t, 0)$, for $0 < t < 1$ defines the curve shown. You could imagine it as a ball moving 1 unit on the x-axis from $t = 0$ to $t = 1$. Without doing any mathematics, you could intuitively see that the velocity is going to be constant and so acceleration would be 0.

Mathematically, we can take the derivative of $r(t)$ by taking the derivative of its components. So we would have $r'(t) = (1, 0)$. This would be the velocity of the ball, a constant, unit vector pointing towards the x-axis. The acceleration, as you can guess, would be $r''(t) = (0, 0)$ which is the 0 vector.

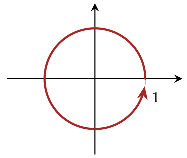


Parabola. The function $r(t) = (3t, 4t - \frac{9.81}{2}t^2)$ for $0 < t < 1$ defines the curve shown. You could imagine it as throwing a ball at the origin under the effect of gravity. Could you try to understand this motion by considering the coordinates separately and using the equations we developed in Part I?



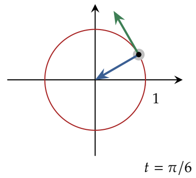
Intuitively, we know that the velocity would have the same x-component for all time t and that the acceleration would be a constant vector pointing downwards.

Mathematically, we have $r'(t) = (3, 4 - 9.81t)$ and $r''(t) = (0, -9.81)$. In the figure, the green vector is the velocity and the blue vector is the acceleration. Both vectors' magnitude are scaled down by a factor of $1/3$.



Circle. The function $r(t) = (\cos t, \sin t)$ for $0 < t < 2\pi$ defines the curve shown. You can imagine as a ball uniformly rotating around the origin with radius 1.

Intuitively, we know that the velocity would be the tangent vector to the circle. The magnitude would be constant (1) as the motion is uniform. Acceleration would also be constant and pointing towards the origin.



Mathematically, we have $r'(t) = (-\sin t, \cos t)$ and $r''(t) = (-\cos t, -\sin t)$, which aligns with our intuition. Once again the green vector is the velocity and the blue vector is the acceleration.

As you can see the amount of behaviour we can model with curves (the explicit construction of the $r(t)$ function is called curve parameterization) is highly unconstrained! It is powerful enough to describe a far wider range of curves than just plots of $y = f(x)$ (which one can imagine as plotting $r(t) = (t, f(t))\ \forall t \in \mathbb{R}$). There are other ways of constructing curves such as using level sets.

Projectile Motion

Similar to part I, the crucial assumption in DSE projectile motion is that the only force exerted on the particle is the gravitational force. So once again we have

$$\begin{aligned} r''(t) &= (0, -g) \\ r'(t) &= (\dot{C}_1, -gt + C_1) \\ r(t) &= (\dot{C}_1 t + \dot{C}_2, -\frac{1}{2}gt^2 + C_1 t + C_2) \end{aligned}$$

for some constants $\dot{C}_1, \dot{C}_2, C_1, C_2$ by repeated "integration". Similar to part I we could find those constants in terms of initial velocities/displacement. As such most properties of projectile motion could be analysed by splitting into x and y-axis.

Another way of looking at it courtesy of Henry Yip [10] would be to consider

[10] https://henry-yip.github.io/

$$r(t) = (\dot{C}_2, C_2) + (\dot{C}_1, C_1)t + (0, -g/2)t^2$$

which tells you that for small t, $r(t)$ looks like a straight line starting from initial displacement $(\dot{C}_2, C_2)$ with the direction of initial velocity $(\dot{C}_1, C_1)$. Gradually the quadratic term dominates and we get the parabolic shape. This idea is similar to Taylor expansions.

Perhaps, then, the most interesting aspect about projectile motion is the conservation of energy. Why is it that energy is still conserved when we use the magnitude of the velocity vector in kinetic energy (instead of one dimensional velocity)? How does the formalism developed in part I relate to the 2 dimensional case? Let's make some definitions first.

Let $r(t) = (x(t), y(t))$. So $x(t), y(t)$ are the x and y components of $r(t)$ respectively. As such we have $r'(t) = (x'(t), y'(t))$ and $r''(t) = (x''(t), y''(t))$. Now we have

$$\begin{aligned} \text{Kinetic energy } &= T = \frac{1}{2}m(x'(t)^2 + y'(t)^2) \\ \text{Potential energy } &= V = mgy(t) \end{aligned}$$

So we have

$$\begin{aligned} T + V &= \frac{1}{2}m(x'(t)^2 + y'(t)^2) + mgy(t) \\ &= \frac{1}{2}m(\dot{C}_1^2 + (-gt + C_1)^2) + mg\left(-\frac{1}{2}gt^2 + C_1 t + C_2\right) \\ &= m\left[\frac{1}{2}\dot{C}_1^2 + \frac{1}{2}g^2t^2 - gtC_1 + \frac{1}{2}C_1^2 - \frac{1}{2}g^2t^2 + gC_1 t + gC_2\right] \\ &= \frac{1}{2}m(\dot{C}_1^2 + C_1^2) + mgC_2 \end{aligned}$$

which is the total energy at initial time.

A more proper way of doing this would involve multivariable calculus. Again refer to the dynamics lecture notes for a more general analysis on conservative forces.

Uniform Circular Motion

Let's think about a ball uniformly rotating around the origin. We know that two variables completely determine its behaviour, its radius and its velocity. As such we can parameterize $r(t) = (R\cos(kt), R\sin(kt))$ where R is the radius and k is some variable that as it turns out is related to angular velocity.

To intuitively see why k is related to angular velocity: Consider how $r(t) = (\cos(t), \sin(t)), 0 < t < 2\pi$ is one full anticlockwise rotation around the unit circle, but $r(t) = (\cos(2t), \sin(2t)), 0 < t < \pi$ is the same full anticlockwise rotation in half the time. We doubled k and the time taken is halved. Could you guess a relationship between k and angular velocity before we do the maths?

Let's find out the velocity and the acceleration. We have

$$\begin{aligned} r(t) &= (R\cos(kt), R\sin(kt)) \\ r'(t) &= (-Rk\sin(kt), Rk\cos(kt)) \\ r''(t) &= (-Rk^2\cos(kt), -Rk^2\sin(kt)) \end{aligned}$$

These formulae immediately tell us all we know about uniform circular motion!

Firstly, $r(t) \perp r'(t) \perp r''(t)$ from simple coordinate geometry (or you could use the dot product if you are familiar with linear algebra).

Secondly, the magnitude of the velocity is

$$\sqrt{R^2k^2(\sin^2(kt) + \cos^2(kt))} = Rk.$$

So we now know $v = Rk$.

What about angular velocity? We see that for a full anticlockwise rotation to take place, t needs to go from 0 to $2\pi/k$. The total angular change would be 2π. As such the angular velocity is $\frac{2\pi k}{2\pi} = k$. So k is the angular velocity!

Finally, $r''(t) = -k^2 r(t)$, so $a = k^2 R$!

As such we also have $a = v^2/R$.

Part III: Waves

In the final part, we would discuss waves. How do we formulate waves mathematically? Why are waves often depicted by sine curves?

Wave Equation

The wave equation is the (partial differential) equation that describes all sorts of waves (water, sound, light ...) It can be

written compactly as

$$\ddot{u} = c^2\nabla^2 u.$$

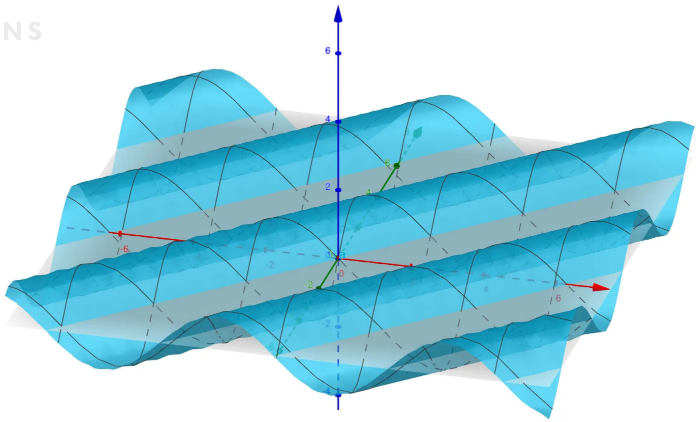
Unfortunately, to understand and derive the above would involve heavy calculus, even if we confine ourselves to one-dimensional waves.

Instead, we would like to explore the mathematical formulation of the sinusoidal travelling wave. The sinusoidal travelling wave is one of many solutions to the wave equation and is the one studied extensively in DSE physics.

The one-dimensional sinusoidal travelling wave could be represented by $u(t,x) = A\sin(kx - wt + \psi)$ where x is distance and t is time for some constants A, k, w, ψ. Try guessing what physical meaning those constants have! It would be revealed at the end.

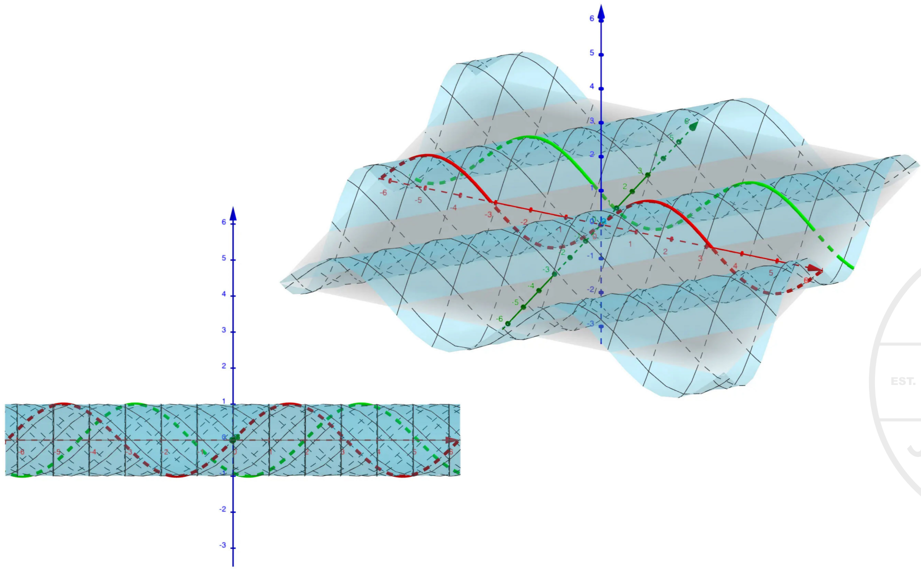
You could imagine this as a function from $\mathbb{R}^2$ to $\mathbb{R}$. It takes in time and distance and tells you the displacement of the wave.

In the graph below, we took $A = w = k = 1$ and $\psi = 0$. The x-axis is red and the t-axis is green.



We can look at how the wave looks like at time c by considering the intersection of the graph $u(t,x)$ and the plane $t = c$.

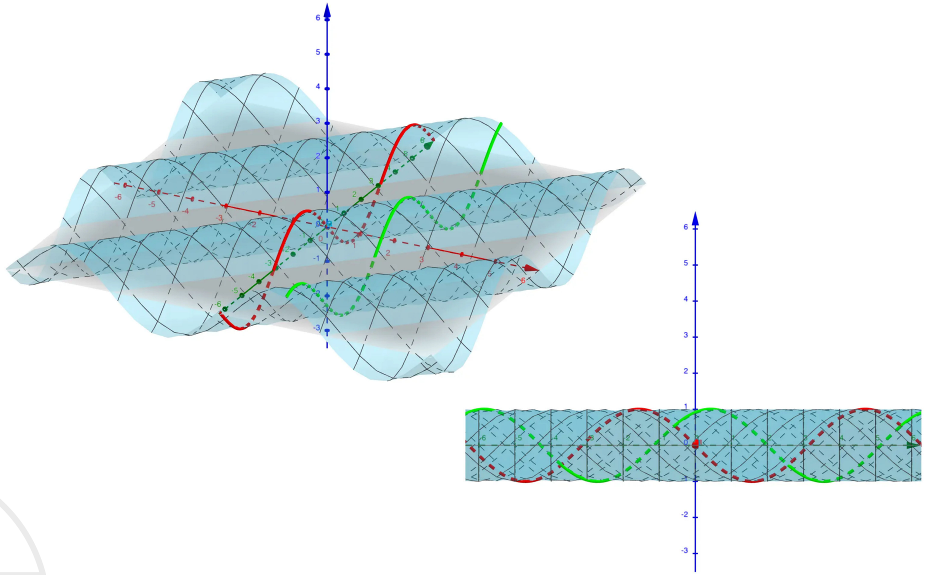
In the graph below, we took $t = 0$ for the red curve and $t = 2$ for the green curve.



If we look at the graph along the time axis, does this look like displacement time graphs? Can you guess what direction the wave is travelling to? How could we change the direction of the wave? What is the wavelength? How does the wavelength correspond to the constants?

Similarly, if we're interested at a particular distance $x = c$, we could look at the intersection of the graph $u(t,x)$ and the plane $x = c$.

In the following graph, we took $x = 0$ for the red curve and $x = 2$ for the green curve.



If we look at the graph along the distance axis, does this look like displacement distance graphs? What is the period of the wave? How does the period correspond to the constants?

Answers and More Questions

Turns out, the wavelength λ is equal to $1/k$ and w is the (angular) frequency of the wave. Can you deduce why that is the case?

What about ψ? What does it represent?

Can you think of how to parametrize stationary waves using a similar $u(x,t)$?

Further Reading

For more on one-dimensional wave equations, there is a LibreTexts article[11] which explains more.

[11] https://chem.libretexts.org/Courses/Pacific_Union_College/Quantum_Chemistry/2%3A_The_Classical_Wave_Equation/.01%3A_The_One-Dimensional_Wave_Equation

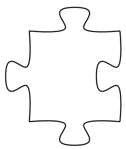
Solving Jigsaws: Start Small, Think Big (Theta)

BY RAPHAEL LI

Introduction

Big jigsaws

Plain-coloured puzzles perplex people, but please passionate polymaths. Unfortunately, I am not one of those polymaths, so solving them remains a tedious task to me. Most jigsaw puzzles have recognizable images printed on them, so as to provide visual clues to how the pieces are to be assembled. Monochromatic jigsaws, however, have none, leaving the puzzled puzzler with what looks more like a white pyramid of snow than a pile of jigsaw pieces.



A puzzle piece with three outies and one innie.

Advertised as "Pure White Hell" puzzles, these tintless knick-knacks are similar to normal jigsaws in that they both consist of square-shaped pieces with *innies* and *outies* on their edges. They must be interlocked together to form a perfect rectangular grid, for which there is only one correct arrangement. However, the solid-coloured version is significantly more challenging than its polychromatic counterpart, as there is absolutely no indication as to how the pieces are meant to fit together other than the shapes of the pieces themselves.

In this article, we will analyze how the difficulty of such hellish puzzles increases with the puzzle's proportions. But to do that, we'll also have to incorporate some basic knowledge of big Θ.

Big **Θ**

Commonly used in asymptotics and computer science, big Θ notation [12] is a way to denote the complexity of a problem at hand [13]. The concept of big Θ notation can be defined as follows.

Definition 1. Let $f(n)$ be the average time taken to solve a problem of order n, where n is a positive integer. If there exists a function $g(n)$ as well as positive constants M_1, M_2 and n_0 such that $0 \leqslant M_1 g(n) \leqslant f(n) \leqslant M_2 g(n)$ for all values of $n \geqslant n_0$, then we say that $f(n) = \Theta(g(n))$.

As an example, consider an algorithm whose runtime can be expressed as $f(n) = 2.023n^3 + 1.875n + 1.48$, where n is the number of inputs provided. When $n \geqslant 1$,

$$\begin{aligned} 0 \leqslant 2.023n^3 &\leqslant 2.023n^3 + 1.875n + 1.48 \\ &\leqslant 2.023n^3 + 1.875n^3 + 1.48n^3 \end{aligned}$$

Therefore,

$$0 \leqslant 2.023n^3 \leqslant f(n) \leqslant 5.378n^3 \qquad (\forall n \geqslant 1)$$

By taking $M_1 = 2.023$, $M_2 = 5.378$ and $n_0 = 1$, we can show that $f(n) = \Theta(n^3)$. In other words, this function has a big Θ of n^3 and its value increases with the cube of the input size.

[12] Big Θ is pronounced "big theta", with Θ being the eighth letter of the Greek alphabet. Although big Θ notation is often discussed together with big O and big Ω ("omega") notation, only properties of big Θ will be explored in this article.

[13] For the purpose of this article, we will only consider average time complexity, i.e. how the average runtime of an algorithm increases with the number of inputs or the size of the problem. "Best case" and "worst-case" analyses will not be conducted.

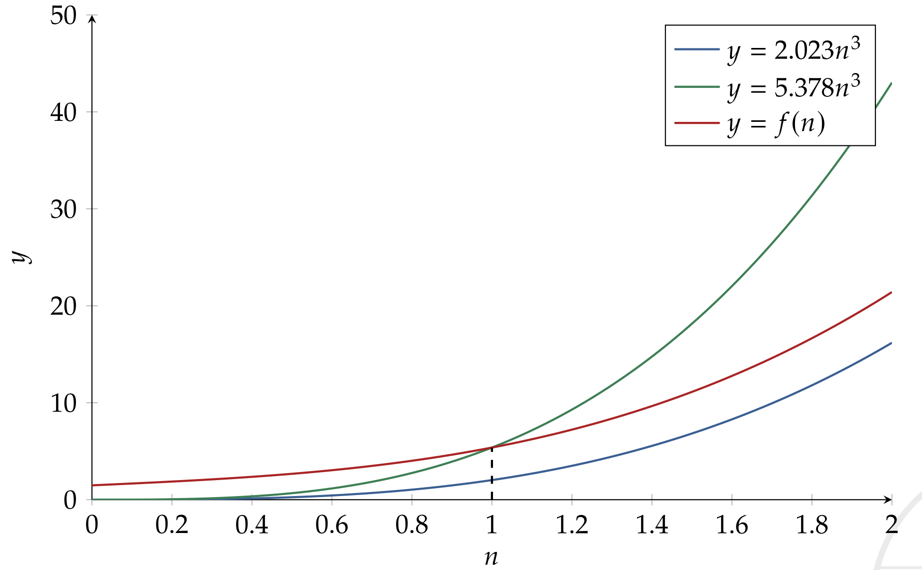


Figure 1. Graphs of $y = f(n)$, $y = 2.023n^3$ and $y = 5.378n^3$. Note how for values of $n > 1$ the graph of $y = f(n)$ is sandwiched between the other two graphs.

Preliminaries

Details of the Problem

Before diving deep into the calculations, let us first make a couple of assumptions about the problem we are dealing with.

For starters, to make matters neat and simple, all jigsaws analyzed in this article will take the form of a square grid and maintain a length-to-width ratio of 1 : 1. We will represent the side length of such a puzzle as a positive integer $n > 2$. We will also denote the total number of pieces as N. Thus, the following relationship can be established:

$$N = n^2 \tag{1}$$

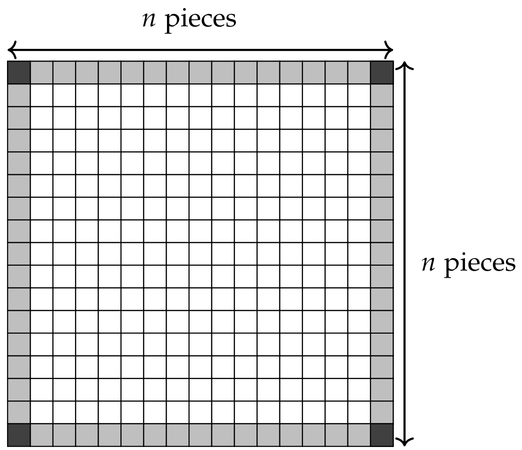
Pieces in a jigsaw puzzle can be divided into three different categories by shape: *corner*, *edge* and *interior*. We will represent the number of corner, edge and interior pieces as c, e and i respectively, all of which can expressed in terms of n as follows.

$$c = 4 \tag{2}$$

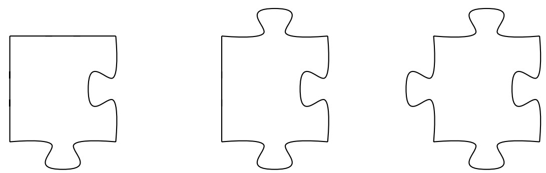
$$e = 4n - 8 \tag{3}$$

$$i = (n-2)^2 = n^2 - 4n + 4 \tag{4}$$

Notice that c remains constant regardless of the puzzle's proportions, because a square grid always has four vertices.



A $n \times n$ square grid consists of 4 corner pieces (dark grey), $(4n-8)$ edge pieces (light grey) as well as $(n-2)^2$ interior pieces (white). Innies and outies not shown.



From left to right: a corner piece, an edge piece and an interior piece.

Theorems Regarding Big $\boldsymbol{\Theta}$ *Notation*

In this subsection, we will introduce and prove several theorems involving big Θ notation. Since most of these theorems are to be applied in the context of analyzing algorithms, the functions involved (which usually represent runtimes of algorithms) can be assumed to be positive-valued.

Theorem 1. *Let C be a positive constant. If $f(n) = \Theta(g(n))$, then $C \cdot f(n) = \Theta(g(n))$.*

Proof. Since $f(n) = \Theta(g(n))$, we have

$$\begin{aligned} 0 \leqslant M_1 g(n) &\leqslant f(n) \leqslant M_2 g(n) \\ 0 \leqslant C \cdot M_1 g(n) &\leqslant C \cdot f(n) \leqslant C \cdot M_2 g(n) \end{aligned}$$

for all values of $n \geqslant n_0$, where M_1, M_2 and n_0 are positive constants.

Suppose $M_3 = C \cdot M_1$ and $M_4 = C \cdot M_2$. It follows that

$$0 \leqslant M_3 g(n) \leqslant C \cdot f(n) \leqslant M_4 g(n). \qquad (\forall n \geqslant n_0)$$

Therefore, $C \cdot f(n) = \Theta(g(n))$. □

Theorem 2. *If $f_1(n) = \Theta(g(n))$ and $f_2(n) = \Theta(g(n))$, then $f_1(n) + f_2(n) = \Theta(g(n))$.*

Proof. From given,

$$\begin{aligned} 0 \leqslant M_1 g(n) \leqslant f_1(n) \leqslant M_2 g(n) \qquad (\forall n \geqslant n_0) \\ 0 \leqslant M_3 g(n) \leqslant f_2(n) \leqslant M_4 g(n) \qquad (\forall n \geqslant n_1) \end{aligned}$$

where M_1, M_2, M_3, M_4, n_0 and n_1 are positive constants. Without loss of generality, assume $n_0 \leqslant n_1$.

It follows that

$$0 \leqslant (M_1 + M_3) g(n) \leqslant f_1(n) + f_2(n) \leqslant (M_2 + M_4) g(n). \qquad (\forall n \geqslant n_1)$$

Therefore, $f_1(n) + f_2(n) = \Theta(g(n))$. □

Theorem 3. *If*

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = C$$

where C is a nonzero finite constant, then $f(n) = \Theta(g(n))$.

Proof. By definition of a limit towards infinity, for every $\epsilon > 0$, there exists a certain positive value n_0 such that whenever $n > n_0$, the inequality

$$\left| \frac{f(n)}{g(n)} - C \right| < \epsilon \qquad (5)$$

holds. If we also add the restraint that $\epsilon < C$, then Inequality 5 can be rewritten as

$$\begin{aligned} 0 < C - \epsilon < \frac{f(n)}{g(n)} < C + \epsilon \\ 0 < (C - \epsilon) g(n) < f(n) < (C + \epsilon) g(n). \end{aligned}$$

Taking $M_1 = C - \epsilon$ and $M_2 = C + \epsilon$ gives

$$0 < M_1 g(n) < f(n) < M_2 g(n) \qquad (\forall n > n_0)$$

which completes the proof. □

Theorem 4. *If $P(n)$ is a polynomial function of degree k, then $P(n) = \Theta(n^k)$.*

Proof. Suppose $P(n) = a_0 n^k + a_1 n^{k-1} + a_2 n^{k-2} + \cdots + a_{k-2} n^2 + a_{k-1} n + a_k$, where k is a positive integer and $a_0, a_1, a_2, \cdots, a_k$ are constants with $a_0 \neq 0$. Notice that

$$\begin{aligned} \lim_{n \to \infty} \frac{P(n)}{n^k} &= \lim_{n \to \infty} \left(a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots + \frac{a_{k-2}}{n^{k-2}} + \frac{a_{k-1}}{n^{k-1}} + \frac{a_k}{n^k} \right) \\ &= \lim_{n \to \infty} (a_0 + 0 + 0 + \cdots + 0 + 0 + 0) \\ &= a_0 \neq 0. \end{aligned}$$

By Theorem 3, $P(n) = \Theta(n^k)$. □

Note that while Theorem 3 holds, its converse does not. For instance, consider the function $f(n) = 2 + \sin n$. Since

$$0 \leqslant 1 \cdot 1 \leqslant f(n) \leqslant 3 \cdot 1,$$

it is apparent that $f(n) = \Theta(1)$. However, the limit

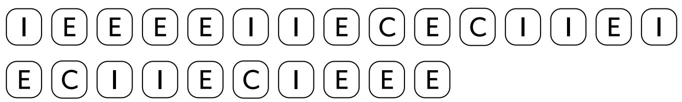
$$\lim_{n\to\infty} \frac{f(n)}{1} = \lim_{n\to\infty} (2 + \sin n)$$

does not exist.

Step One: Sorting Pieces

The first step of our puzzle-solving procedure is *sorting*: the classification of puzzle pieces into corner pieces, edge pieces and interior pieces. This will make the assembly more organized, systematic and efficient later.

Imagine a conveyor belt carrying all the puzzle pieces in a linear fashion, and the puzzler starts "identifying" them one at a time, from left to right. Suppose it takes one second to identify if a certain puzzle piece is a corner, edge or interior one. Since there are a total of N pieces, it may be tempting to jump to the conclusion that sorting all of them will take exactly N seconds, going through and examining every single piece in the pile.



A sequence of corner (C), edge (E) and interior (I) puzzle pieces, arranged in a random order. How long will the sorting process take?

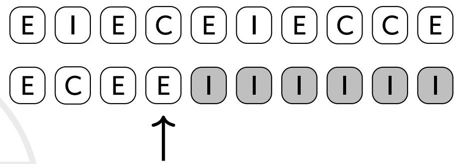
Fortunately, there is a relatively quicker method. Recall that the number of each type of puzzle pieces is known: there are 4 corner pieces, $(14n - 8)$ edge pieces and $(12n^2 - 14n + 4)$ interior pieces. This means that if at some point in

the sorting process we've already picked out the entirety of edge and corner pieces, we can be certain that the rest must all be interior pieces — no further identification required.

If so, what is the expected time needed to sort out all N pieces? It's definitely shorter than N seconds, sure, but by how much?

A Concrete Example

For starters, let us consider the following scenario:



Another random sequence of 25 jigsaw pieces (4 corner pieces, 12 edge pieces and 9 interior pieces). Note the six shaded consecutive interior pieces at the end.

As we can see, the sequence ends in a consecutive run of six interior pieces. What this means is that if we start sorting the pieces from left to right, by the time we finish dealing with the 19th piece (marked with an arrow in the figure above), it will no longer be necessary for us to check any of the remaining pieces — we've already successfully identified four corner pieces and twelve edge pieces, so the rest must be interior.

If we denote the length of the consecutive run at the end as L_i, the duration of the sorting process T_s (in seconds) can be expressed as:

$$\begin{aligned} T_s &= N - L_i \\ &= 25 - 6 \\ &= 19 \end{aligned} \tag{6}$$

which is clearly less than 25 seconds.

Calculating the Expected Duration

We start by taking note of the fact that

$$P(\text{Sequence ends in a run of corner pieces}) = \frac{c}{N}$$
$$P(\text{Sequence ends in a run of edge pieces}) = \frac{e}{N}$$
$$P(\text{Sequence ends in a run of interior pieces}) = \frac{i}{N}$$

because whether each of these events happen is completely dependent on what the very last piece of the sequence is: if the sequence ends in an interior piece, then it must end in a run of interior pieces (whose length is at least 1), and so on.

Meanwhile, the expected length of that run varies depending on the type of puzzle piece involved. For instance, since the sequence comprises of i interior pieces and $(c+e)$ non-interior pieces, the i interior pieces must go into $(c+e+1)$ different consecutive runs of interior pieces (although some may have a length of 0). Therefore, the expected length of such a run can be expressed as

$$E(L_i) = \frac{i}{c+e+1}$$

Similarly, the expected length of a consecutive run of edge pieces $E(L_e)$ and that of corner pieces $E(L_c)$ are expressed as:

$$E(L_e) = \frac{e}{i+c+1}$$
$$E(L_c) = \frac{c}{i+e+1}$$

Lastly, by generalizing Equation 6, the expected period of time $E(T_s)$ needed to complete the sorting process can be

calculated as follows:

$$\begin{aligned}
E(T_s) &= N - E(\text{Length of final consecutive run}) \\
&= N - \left(\frac{c}{N}\cdot E(L_c) + \frac{e}{N}\cdot E(L_e) + \frac{i}{N}\cdot E(L_i)\right) \\
&= N - \left(\frac{c}{N}\cdot\frac{c}{i+e+1} + \frac{e}{N}\cdot\frac{e}{i+c+1} + \frac{i}{N}\cdot\frac{i}{c+e+1}\right) \\
&= N - \left(\frac{c^2}{N(i+e+1)} + \frac{e^2}{N(i+c+1)} + \frac{i^2}{N(c+e+1)}\right) \\
&= N - \frac{1}{N}\left(\frac{c^2}{i+e+1} + \frac{e^2}{i+c+1} + \frac{i^2}{c+e+1}\right) \\
&= N - \frac{1}{N}\left(\frac{c^2}{N-c+1} + \frac{e^2}{N-e+1} + \frac{i^2}{N-i+1}\right)
\end{aligned}$$

By substituting Equations 1 through 4, we have

$$\begin{aligned}
&E(T_s) \\
&= n^2 - \frac{1}{n^2}\left(\frac{4^2}{n^2-4+1} + \frac{(4n-8)^2}{n^2-(4n-8)+1} + \frac{(n-2)^4}{n^2-(n^2-4n+4)+1}\right) \\
&= n^2 - \frac{1}{n^2}\left(\frac{16}{n^2-3} + \frac{16(n-2)^2}{n^2-4n+9} + \frac{(n-2)^4}{4n-3}\right) \qquad (7)
\end{aligned}$$

After a fair amount of computation, this expression comes out as follows.

$$E(T_s) = \frac{4(n-2)(n^2+1)\left(n^6-3n^5+5n^4+7n^3-8n^2+42n-36\right)}{n^2(4n-3)(n^2-3)(n^2-4n+9)} \qquad (8)$$

Applying Big **Θ**

To find the big Θ of the nightmare-inducing algebraic fraction obtained in Equation 8, notice that the numerator and denominator have degrees 9 and 7 respectively. Intuitively, we would expect the function to have a similar growth rate to that of a quadratic function.

To prove this, we will consider the following limit.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{E(T_s)}{n^2} &= \lim_{n\to\infty} \frac{4(n-2)\left(n^2+1\right)\left(n^6-3n^5+5n^4 +7n^3-8n^2+42n-36\right)}{n^4(4n-3)(n^2-3)(n^2-4n+9)} \\
&= \lim_{n\to\infty} \frac{4(1-2/n)\left(1+1/n^2\right)\left(1-3/n+5/n^2 +7/n^3-8/n^4+42/n^5-36/n^6\right)}{(4-3/n)(1-3/n^2)(1-4/n+9/n^2)} \\
&= \lim_{n\to\infty} \frac{4(1-0)(1+0)(1+0)}{(4-0)(1-0)(1+0)} \\
&= 1 \qquad (9)
\end{aligned}
$$

Alternatively, the same limit can also be evaluated using the representation of $E(T_s)$ in Equation 7.

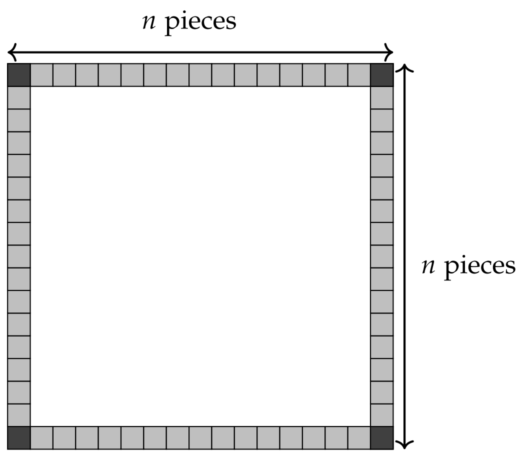
$$
\begin{aligned}
\lim_{n\to\infty} \frac{E(T_s)}{n^2} &= \lim_{n\to\infty} \frac{1}{n^2}\left(n^2 - \frac{1}{n^2}\left(\frac{16}{n^2-3} + \frac{16(n-2)^2}{n^2-4n+9} + \frac{(n-2)^4}{4n-3}\right)\right) \\
&= \lim_{n\to\infty} \left(1 - \frac{1}{n^4}\left(\frac{16}{n^2-3} + \frac{16(n-2)^2}{n^2-4n+9} + \frac{(n-2)^4}{4n-3}\right)\right) \\
&= \lim_{n\to\infty} \left(1 - \frac{16}{n^6-3n^4} - \frac{16(n-2)^2}{n^6-4n^5+9n^4} - \frac{(n-2)^4}{4n^5-3n^4}\right) \\
&= \lim_{n\to\infty} \left(1 - \frac{16n^{-6}}{1-3n^{-2}} - \frac{16\left(1-2n^{-1}\right)^2 n^{-4}}{1-4n^{-1}+9n^{-2}} - \frac{\left(1-2n^{-1}\right)^4 n^{-1}}{4-3n^{-1}}\right) \\
&= \lim_{n\to\infty} \left(1 - \frac{16n^{-6}}{1-3n^{-2}} - \frac{16\left(1-2n^{-1}\right)^2 n^{-4}}{1-4n^{-1}+9n^{-2}} - \frac{\left(1-2n^{-1}\right)^4 n^{-1}}{4-3n^{-1}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left(1 - \frac{0}{1-0} - \frac{0}{1-0+0} - \frac{0}{4-0}\right) \\
&= 1
\end{aligned}
$$

As shown in both calculations, when n approaches infinity, $E(T_s)/n^2$ gets closer and closer to a finite nonzero constant: 1. By Theorem 3, the rational function $E(T_s)$ has a big Θ of n^2.

Step Two: Constructing a Frame

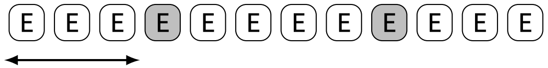
After sorting our pieces by shape, the next step is to construct a square frame by connecting all the edge and corner pieces. Again, we'll assume that it takes us exactly one second to determine whether a specific puzzle piece fits and can be inserted in a certain position.



The frame of an $n \times n$ square grid, consisting of only edge and corner pieces.

The First Connection

We start with a randomly selected corner piece. There are two *target* edge pieces that can be connected to this corner piece validly, and successfully picking any one of those two will take, on average, $(e+1)/3$ seconds. This is because in a randomly arranged sequence of e edge pieces, the expected length of each consecutive run of non-target pieces is $(e-2)/3$.



$\approx (e-2)/3$

A random sequence of 12 edge pieces, two of which are considered the "target" (shaded).

Therefore, the first target piece will be found at position

$$\frac{e-2}{3} + 1 = \frac{e+1}{3}.$$

The Rest of the First Edge

Out of the remaining $(e-1)$ edge pieces, only one can be successfully linked to the piece we've just attached. Searching for this one piece takes about $(e-1)/2$ seconds on average.

As the pool of candidates narrows down, the time needed to locate each of the pieces in the rest of the edge will take less and less time.

$$\begin{aligned} \text{Time needed for the 2nd edge piece} &= (e-1)/2 \\ \text{Time needed for the 3rd edge piece} &= (e-2)/2 \\ &\vdots \\ \text{Time needed for the } (e/4)\text{th edge piece} &= (e-(e/4-1))/2 \end{aligned}$$

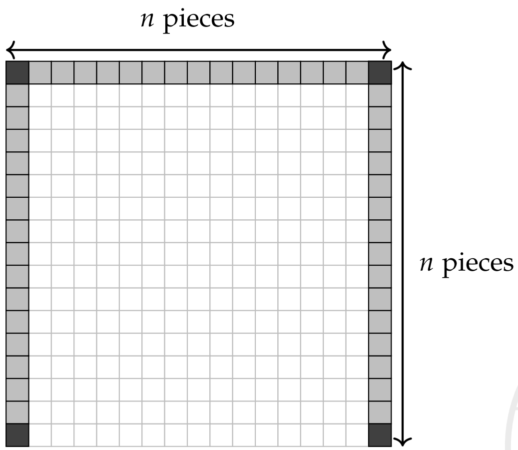
To top it all off, we will finish the edge by attaching the second corner piece, looking for which will take an average of 2 seconds.

Taking all into account, finishing the first edge of the puzzle will require a total of:

$$\begin{aligned} &\frac{e+1}{3} + \left(\sum_{k=1}^{e/4-1} \frac{1}{2}(e-k)\right) + 2 \\ &= \frac{e+1}{3} + \frac{1}{2}\left(\sum_{k=1}^{e/4-1} e - \sum_{k=1}^{e/4-1} k\right) + 2 \\ &= \frac{e+1}{3} + \frac{1}{2}\left(e\left(\frac{e}{4}-1\right) - \frac{1}{2}\cdot\frac{e}{4}\cdot\left(\frac{e}{4}-1\right)\right) + 2 \\ &= \frac{e+1}{3} + \frac{e}{2}\cdot\left(\frac{e}{4}-1\right)\left(1-\frac{1}{2}\cdot\frac{1}{4}\right) + 2 \\ &= \frac{e+1}{3} + \frac{7e}{16}\cdot\left(\frac{e}{4}-1\right) + 2 \\ &= \frac{e+1}{3} + \frac{7e^2}{64} - \frac{7e}{16} + 2 \\ &= \frac{21e^2 - 20e + 448}{192} \quad \text{seconds.} \end{aligned}$$

Constructing the Adjacent Edges

After constructing the first edge of the puzzle, we'll now proceed to the two edges perpendicular to the existing one.



Building the second and third edge of the puzzle.

As only $3e/4$ edge pieces remain, the amount of time required to build the second edge should be relatively shorter. Specifically, this process will take

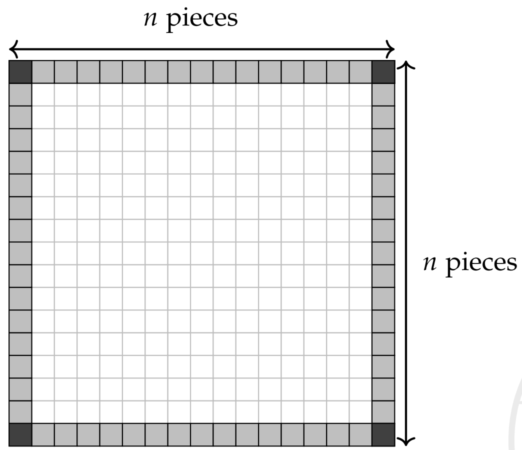
$$\begin{aligned}
&\left(\sum_{k=0}^{e/4-1} \frac{1}{2}\left(\frac{3e}{4} - k\right)\right) + \frac{1}{2} + 1 \\
&= \frac{1}{2}\left(\sum_{k=0}^{e/4-1} \frac{3e}{4} - \sum_{k=0}^{e/4-1} k\right) + \frac{1}{2} + 1 \\
&= \frac{1}{2}\left(\frac{3e^2}{16} - \frac{1}{2} \cdot \frac{e}{4} \cdot \left(\frac{e}{4} - 1\right)\right) + \frac{3}{2} \\
&= \frac{1}{2}\left(\frac{3e^2}{16} - \frac{e^2}{32} + \frac{e}{8}\right) + \frac{3}{2} \\
&= \frac{5e^2 + 4e + 96}{64} \quad \text{seconds.}
\end{aligned}$$

The same goes to the third edge — the only difference will be the number of remaining pieces. By modifying the above calculation, the time needed to construct this third edge can be represented as

$$\begin{aligned}
&\left(\sum_{k=0}^{e/4-1} \frac{1}{2}\left(\frac{e}{2} - k\right)\right) + 1 \\
&= \frac{1}{2}\left(\sum_{k=0}^{e/4-1} \frac{e}{2} - \sum_{k=0}^{e/4-1} k\right) + 1 \\
&= \frac{1}{2}\left(\frac{e^2}{8} - \frac{1}{2} \cdot \frac{e}{4} \cdot \left(\frac{e}{4} - 1\right)\right) + 1 \\
&= \frac{1}{2}\left(\frac{e^2}{8} - \frac{e^2}{32} + \frac{e}{8}\right) + 1 \\
&= \frac{3e^2 + 4e + 64}{64} \quad \text{seconds.}
\end{aligned}$$

The Last Edge

At this point, only $e/4$ edge pieces remain, all of which will go into the fourth (and last) edge of the puzzle's frame.



Building the fourth edge of the puzzle.

As expected, the calculation here is akin to those in the previous subsection; completing the last part of the frame will take, on average,

$$\begin{aligned}
&\sum_{k=0}^{e/4-1} \frac{1}{2}\left(\frac{e}{4} - k\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^{e/4-1} \frac{e}{4} - \sum_{k=0}^{e/4-1} k\right) \\
&= \frac{1}{2}\left(\frac{e^2}{16} - \frac{1}{2}\cdot\frac{e}{4}\cdot\left(\frac{e}{4} - 1\right)\right) \\
&= \frac{1}{2}\left(\frac{e^2}{16} - \frac{e^2}{32} + \frac{e}{8}\right) \\
&= \frac{e^2 + 4e}{64} \quad \text{seconds.}
\end{aligned}$$

Summing It All Up

In total, the expected duration $E(T_f)$ needed to construct the entire frame of the puzzle can be represented in terms of e as

$$\begin{aligned}
E(T_f) &= \frac{21e^2 - 20e + 448}{192} + \frac{5e^2 + 4e + 96}{64} \\
&\quad + \frac{3e^2 + 4e + 64}{64} + \frac{e^2 + 4e}{64} \\
&= \frac{3e^2 + e + 58}{12}
\end{aligned}$$

Using Equation 3, the same value can be expressed in terms of n as

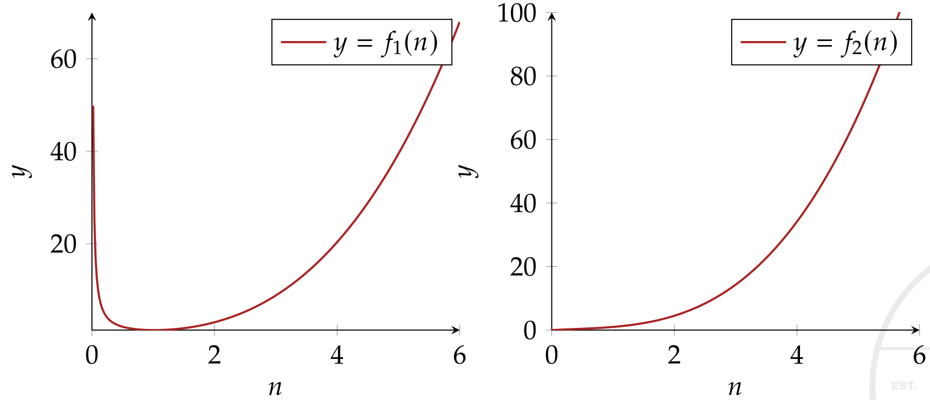
$$\begin{aligned}
E(T_f) &= \frac{3(4n-8)^2 + (4n-8) + 58}{12} \\
&= \frac{48n^2 - 188n + 242}{12} \\
&= \frac{24n^2 - 94n + 121}{6} \qquad (10)
\end{aligned}$$

Applying big $\boldsymbol{\Theta}$

As $E(T_f) = (24n^2 - 94n + 121)/6$ is a polynomial function of degree 2, we can make use of Theorem 4 and conclude that $E(T_f)$ has a big Θ of n^2, which is identical to that of $E(T_s)$, the expression we obtained in the section about sorting. However, just because both runtimes have the same big Θ does not imply that the sorting and frame-building processes take the same amount of time.

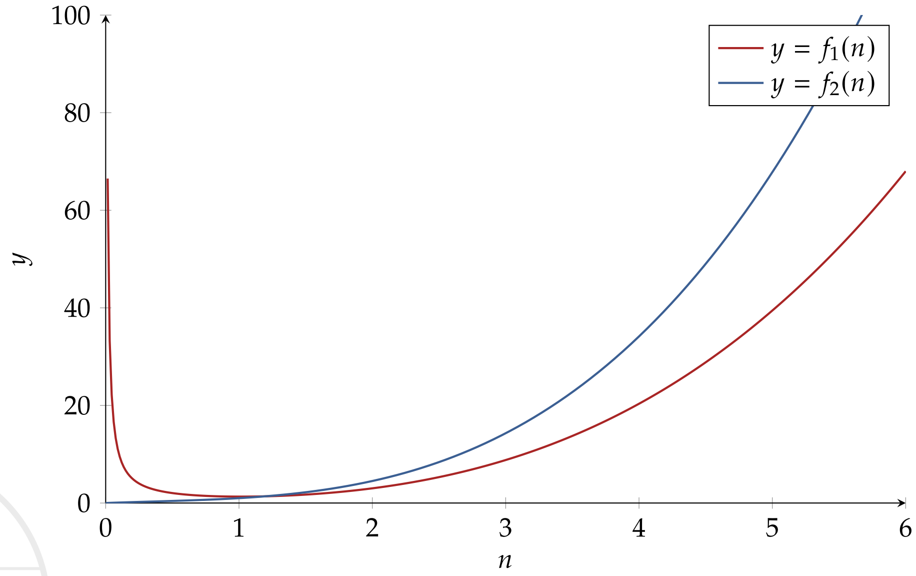
To illustrate this, consider the functions $f_1(n) = 0.314n^3 + 1/n$ and $f_2(n) = 0.628n^3 - 0.628n^2 + n$. By Theorems 3 and 4 respectively, both functions have a big Θ of n^3, which indicates that both of the functions go up with the cube of n.

Indeed, if we plot the two functions on separate coordinate systems and look at the big picture, we can see that although the graph of $f_1(n)$ contains a vertical asymptote of $x = 0$, both graphs have a roughly similar shape. This is characteristic of functions with the same big Θ.



Graphs of $y = f_1(n)$ and $y = f_2(n)$ on separate coordinate systems. Note the different scales used for the y-axes.

Despite this, plotting both functions on the same plane reveals that $f_1(n)$ and $f_2(n)$ are entirely different functions. If $f_1(n)$ and $f_2(n)$ represent the runtimes of two different algorithms, the one with runtime $f_1(n)$ may be considered more efficient, especially for larger values of n.



Graphs of $y = f_1(n)$ and $y = f_2(n)$ on the same coordinate plane.

An even more obvious example of this can be seen in Figure 1. Despite the similar shape of the three cubic graphs, there is a clear difference as to the actual magnitudes of function outputs.

Step Three: Filling Up the Interior

To complete what's left of the puzzle, all we have to do is to assemble the interior pieces. As shown in Equation 4, there are $i = (n-2)^2$ such pieces.

To do this, we will work from left to right and from top to bottom, employing as always the strategy of randomly selecting an unused piece and trying to attach it to the puzzle. If such an operation takes one second, the expected amount of time necessary to complete the hollow and interior part of the puzzle, denoted as $E(T_i)$, can be calculated as follows.

$$
\begin{aligned}
E(T_i) &= \frac{i+1}{2} + \frac{i}{2} + \frac{i-1}{2} + \frac{i-2}{2} + \cdots + 1 \\
&= \sum_{k=0}^{i-1} \frac{i-k+1}{2} \\
&= \frac{1}{2}\left(\sum_{k=0}^{i-1}(i+1) - \sum_{k=0}^{i-1} k\right) \\
&= \frac{1}{2}\left(i(i+1) - \frac{i(i-1)}{2}\right) \\
&= \frac{i^2+3i}{4}
\end{aligned}
$$

Substituting in Equation 4 gives the expected time in terms of n:

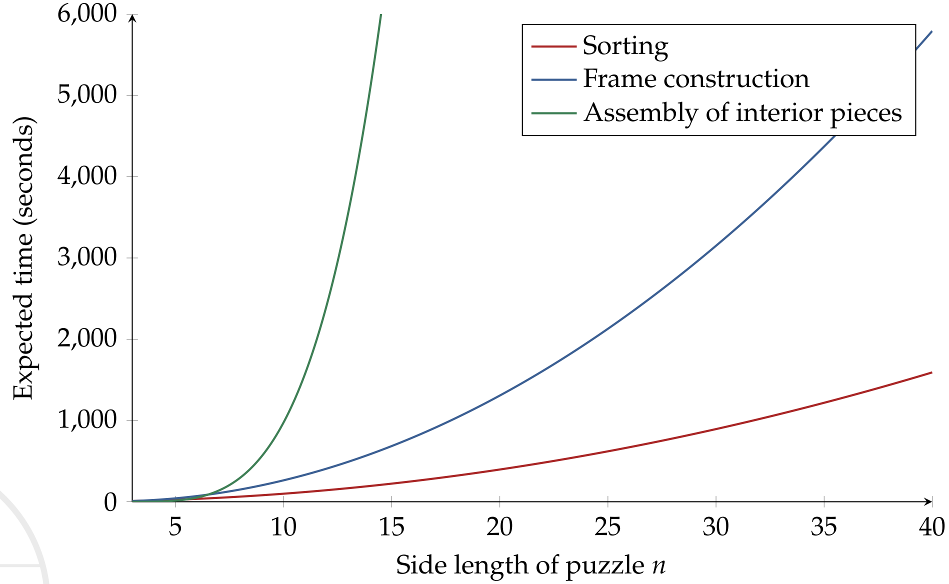
$$E(T_i) = \frac{(n-2)^4 + 3(n-2)^2}{4} \tag{11}$$

which is a polynomial of degree 4 and thus has a big Θ of n^4.

Linking the Three Steps Together

In the last three sections, we determined the amount of time needed to successfully sort and assemble all n^2 pieces in an $n \times n$ jigsaw puzzle. All that's left to do is to add them up to obtain the expected time required $E(T_p)$ to complete the entire jigsaw from start to finish.

$$
\begin{aligned}
E(T_s) &= \frac{4(n-2)\left(n^2+1\right)\left(n^6-3n^5+5n^4+7n^3-8n^2+42n-36\right)}{n^2(4n-3)(n^2-3)(n^2-4n+9)} \\
E(T_f) &= \frac{24n^2-94n+121}{6} \\
E(T_i) &= \frac{(n-2)^4+3(n-2)^2}{4}
\end{aligned}
$$



Expected duration of different parts of the puzzle-solving procedure. Parts of the graphs where $n < 3$ are not plotted.

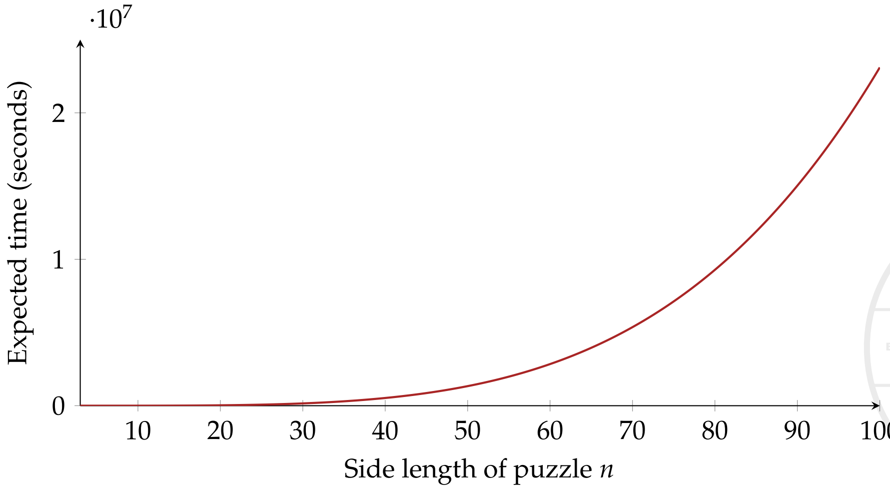
By summing together Equations 7, 8 and 9, we have

$$
\begin{aligned}
E(T_p) = {} & \frac{4(n-2)\left(n^2+1\right)\left(n^6-3n^5+5n^4+7n^3-8n^2+42n-36\right)}{n^2(4n-3)(n^2-3)(n^2-4n+9)} \\
& + \frac{24n^2-94n+121}{6} + \frac{(n-2)^4+3(n-2)^2}{4} \\
= {} & \frac{\begin{array}{l}(12n^{11}-153n^{10}+1056n^9-4115n^8+9148n^7-7109n^6 \\ -16752n^5+54963n^4-63480n^3+26814n^2-5760n+3456)\end{array}}{12n^2(4n-3)(n^2-3)(n^2-4n+9)}.
\end{aligned}
$$

Instead, recall the following conclusions drawn in Sections , and respectively.

$$
\begin{aligned}
E(T_s) &= \Theta(n^2) \\
E(T_f) &= \Theta(n^2) \\
E(T_i) &= \Theta(n^4)
\end{aligned}
$$

Since $E(T_i)$ has the greatest big Θ out of all three, it has the highest growth rate and is thus likely to outgrow $E(T_s)$ and $E(T_f)$ as n gets larger and larger. Therefore, if we were to guess, the function $E(T_p)$ would probably have a big Θ of n^4. This hypothesis is also supported by the fact that the graph for $E(T_p)$ bears a resemblance to that of a quartic function.



Expected duration of solving a jigsaw puzzle. Parts of the graphs where $n < 3$ are not plotted.

Again, we will use Theorem 4 in order to prove this hypothesis.

$$\begin{aligned}
&\lim_{n\to\infty}\frac{E(T_p)}{n^4}\\
&= \lim_{n\to\infty}\frac{E(T_s)}{n^4} + \lim_{n\to\infty}\frac{E(T_f)}{n^4} + \lim_{n\to\infty}\frac{E(T_i)}{n^4}\\
&= \lim_{n\to\infty}\left(\frac{E(T_s)}{n^2}\cdot\frac{1}{n^2}\right) + \lim_{n\to\infty}\frac{E(T_f)}{n^4} + \lim_{n\to\infty}\frac{E(T_i)}{n^4}\\
&= \lim_{n\to\infty}\left(1\cdot\frac{1}{n^2}\right) + \lim_{n\to\infty}\frac{E(T_f)}{n^4} + \lim_{n\to\infty}\frac{E(T_i)}{n^4} \quad \text{(by Equation 9)}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{n\to\infty}\frac{E(T_f)}{n^4} + \lim_{n\to\infty}\frac{E(T_i)}{n^4}\\
&= \lim_{n\to\infty}\frac{24n^2-94n+121}{6n^4} + \lim_{n\to\infty}\frac{(n-2)^4+3(n-2)^2}{4n^4}\\
&= \lim_{n\to\infty}\frac{24n^{-2}-94n^{-3}+121n^{-4}}{6}\\
&+ \lim_{n\to\infty}\frac{(1-2n^{-1})^4+3(1-2n^{-1})^2n^{-2}}{4}\\
&= 0 + \lim_{n\to\infty}\frac{(1+0)^4+0}{4} = \lim_{n\to\infty}\frac{1^4}{4} = \frac{1}{4}
\end{aligned}$$

As we can see, the limit $\lim_{n\to\infty} E(T_p)/n^4$ evaluates to $1/4$, a nonzero and finite constant. Hence, $E(T_p) = \Theta(n^4)$.

Conclusion

As we have shown, the amount of time required to solve a solid-coloured jigsaw puzzle of proportions $n \times n$ can be estimated using the following formula, in which $n \geqslant 3$.

$$E(T_p) = \frac{\begin{array}{c}(12n^{11} - 153n^{10} + 1056n^9 - 4115n^8 + 9148n^7 - 7109n^6\\ -16752n^5 + 54963n^4 - 63480n^3 + 26814n^2 - 5760n + 3456)\end{array}}{12n^2(4n-3)(n^2-3)(n^2-4n+9)}.$$

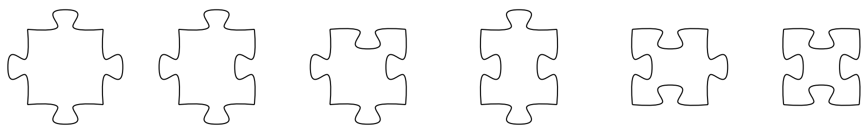
A table overleaf shows several results derived from this formula.

Side length	Average time needed to solve
3	18.6 seconds
10	23.9 minutes, about twice as long as the first spacewalk
20	7.83 hours, roughly half the duration of the first transatlantic flight
50	15.5 days, about 3.5 times as long as the voyage of the Titanic
100	267 days, approximately the length of a full-term pregnancy
200	12.2 years, about the orbital period of Jupiter
500	487 years, about twice the orbital period of Pluto
1000	7.86 millenia, about 1500 times the duration of World War II

The average amount of time needed to solve square-shaped puzzles of various sizes. Results are rounded.

We have also shown that the puzzle-solving procedure has a big Θ of n^4. What this means is that while doubling the side length of a jigsaw puzzle quadruples its area, its difficulty (as reflected by the average time needed to solve it) increases 16-fold, since $16 = 2^4$. In general, an increase in side length by k times will increase the difficulty by a factor of k^4.

It is important to note that this may not accurately reflect the actual, real-life situation. In reality, it may be more practical to further sort the interior pieces into six different groups based on the positions of their innies and outies, as shown in the figure below. In addition to their positions, inspecting the shapes of the innies and outies in detail may also help speed up the process.



From left to right: Examples of jigsaw pieces with 4 outies, 3 outies, 2 adjacent outies, 2 opposite outies, 1 outie and no outies.

Feynman's Intergral Trick

BY ANTHONY LAI

Introduction

Feynman's Trick is a powerful integration technique used to compute Definite Integrals. Initially popularized by American theoretical physicist Richard Feynman (1918-1988), (hence its common name as Feynman's Trick). It twists the basis of the problem by converting the problem from an integral to a differential equation.

Richard Phillips Feynman

Other names for Feynman's Trick include its proper name: The Leibniz Integral Rule, and a more descriptive name of the technique: Differentiation under the Integral Sign.

Basic Idea

The basis of Feynman's Trick is to introduce a new variable into a Definite Integral, differentiating the integral with respect to the new variable, and in the process, simplifying the integral and converting the problem into a differential equation.

For an integral of an arbitrary function $f(x)$, the integral may be transformed by considering a new variable t as follows:

$$\int f(x)\,dx \longrightarrow \text{Let } g(t) = \int f(x,t)\,dx.$$

The next step involves differentiating the new function $g(t)$ with respect to the new variable t

$$g'(t) = \int \frac{\partial}{\partial t}[f(x,t)]\,dx.$$

The simplified on the right-hand side should then be simplified into a function of t, the problem is now a differential equation and can be solved by integrating both sides with respect to t.

Working Example

Suppose you were told to compute the following integral:

$$\int_0^1 \frac{x^3 - 1}{\ln x}\,dx.$$

Usual integration techniques taught in school such as Integration by Substitution, Trigonometric substitutions, and Integration by Parts do not successfully compute a result for this problem. This is where Feynman's Trick comes in handy.

We start by introducing a new variable t into the integral, which involves replacing a parameter in the integral with t and then considering the entire integral to be a function of t.

$$\text{Let } f(t) = \int_0^1 \frac{x^t - 1}{\ln x}\,dx.$$

The next step is to differentiate both sides of the integral

with respect to t.

$$\frac{d}{dt} f(t) = \int_0^1 \frac{d}{dt}\left(\frac{x^t - 1}{\ln x}\right) dx$$
$$f'(t) = \int_0^1 \frac{x^t \ln x}{\ln x}\,dx = \int_0^1 x^t\,dx = \left.\frac{x^{t+1}}{t+1}\right|_0^1 = \frac{1}{t+1}.$$

It should be noted that x and by extension dx should be considered as constants since we are differentiating with respect to t.

Now we have simplified the integral and obtained a result which is essentially a differential equation. We can then integrate both sides of this differential equation with respect to t.

$$\int f'(t)\,dt = \int \frac{1}{t+1}\,dt$$
$$f(t) = \ln|t+1| + C$$
$$\implies \int_0^1 \frac{x^t - 1}{\ln x}\,dx = \ln|t+1| + C$$

We can now write the function $f(t)$ as a regular expression in terms of t, though the integration constant C remains a problem as it is arbitrary. We can eliminate this constant by setting t to a convenient value, e.g. 0 and solving for C:

Let $t = 0$:

$$\int_0^1 \frac{x^0 - 1}{\ln x}\,dx = \ln|0+1| + C$$
$$\int_0^1 \frac{0}{\ln x}\,dx = \ln|1| + C$$
$$0 = 0 + C$$
$$\implies C = 0$$
$$\implies f(t) = \int_0^1 \frac{x^t - 1}{\ln x}\,dx = \ln|t+1|$$

Finally, we can solve the original problem by substituting

$t = 3$:

$$f(3) = \int_0^1 \frac{x^3 - 1}{\ln x}\, dx = \ln|3+1| = \boxed{\ln 4}.$$

This is how Feynman's Trick can be applied to compute Definite Integrals.

Conclusion and Challenge for Readers

Feynman's Trick is quite a different approach to computing integrals compared to the standard methods taught in school. Back in Feynman's day, Feynman apparently built himself a reputation of being a "master" at computing integrals using this trick. Feynman himself described this trick in his autobiography *Surely you're joking, Mr. Feynman*, as part of a "different box of tools" which can be used to tackle integration problems.

To end off our exploration of Mr. Feynman's neat little trick, I challenge you, dear reader, to compute the following integral:

$$\int_0^1 \frac{\ln(1 + x - x^2)}{x}\, dx.$$

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