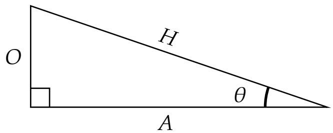
TRIGONOMETRIC FUNCTIONS REFERENCE

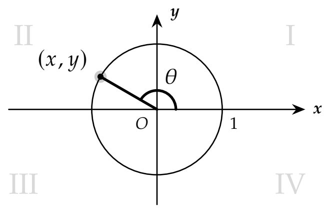
1. Right-Angled Triangle Definition



$(0 < \theta < \pi/2)$

$$\sin\theta = O/H \quad \csc\theta = H/O$$
$$\cos\theta = A/H \quad \sec\theta = H/A$$
$$\tan\theta = O/A \quad \cot\theta = A/O$$

2. Unit Circle Definition



$$\sin\theta = y \quad \csc\theta = 1/y$$
$$\cos\theta = x \quad \sec\theta = 1/x$$
$$\tan\theta = y/x \quad \cot\theta = x/y$$

3. Domain Restrictions

$(n \in \mathbb{Z})$

$\sin\theta$: *none* — $\csc\theta$: $\theta \neq n\pi$

$\cos\theta$: *none* — $\sec\theta$: $\theta \neq (n + 1/2)\pi$

$\tan\theta$: $\theta \neq (n + 1/2)\pi$ — $\cot\theta$: $\theta \neq n\pi$

4. Range

$$-1 \leqslant \sin\theta \leqslant 1 \quad \csc\theta \leqslant -1 \text{ or } 1 \leqslant \csc\theta$$
$$-1 \leqslant \cos\theta \leqslant 1 \quad \sec\theta \leqslant -1 \text{ or } 1 \leqslant \sec\theta$$
$$-\infty < \tan\theta < \infty \quad -\infty < \cot\theta < \infty$$

5. Odd and Even Relations

$$\sin(-\theta) = -\sin\theta \quad \csc(-\theta) = -\csc\theta$$
$$\cos(-\theta) = \cos\theta \quad \sec(-\theta) = \sec\theta$$
$$\tan(-\theta) = -\tan\theta \quad \cot(-\theta) = -\cot\theta$$

6. Square Relations

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$
$$\csc^2\theta + \sec^2\theta = \csc^2\theta\sec^2\theta$$

7. Double Angle Formulae

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$
$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$
$$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

8. Compound Angle Formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\csc(A \pm B) = \frac{\csc A \csc B \sec A \sec B}{\sec A \csc B \pm \csc A \sec B}$$

$$\sec(A \pm B) = \frac{\csc A \csc B \sec A \sec B}{\csc A \csc B \mp \sec A \sec B}$$

$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$$

9. Product-to-Sum Formulae

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$

$$\cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$$

$$\sin A \sin B = -\frac{1}{2}(\cos(A+B) - \cos(A-B))$$

10. Sum-to-Product Formulae

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

11. Derivatives and Integrals

$$\begin{array}{rcccl}
\cos\theta & \xleftarrow{\frac{d}{d\theta}} & \mathbf{\sin\theta} & \xrightarrow{\int} & -\cos\theta + C \\
-\sin\theta & \xleftarrow{\frac{d}{d\theta}} & \mathbf{\cos\theta} & \xrightarrow{\int} & \sin\theta + C \\
\sec^2\theta & \xleftarrow{\frac{d}{d\theta}} & \mathbf{\tan\theta} & \xrightarrow{\int} & \ln|\sec\theta| + C \\
-\csc\theta\cot\theta & \xleftarrow{\frac{d}{d\theta}} & \mathbf{\csc\theta} & \xrightarrow{\int} & \ln|\csc\theta - \cot\theta| + C \\
\sec\theta\tan\theta & \xleftarrow{\frac{d}{d\theta}} & \mathbf{\sec\theta} & \xrightarrow{\int} & \ln|\sec\theta + \tan\theta| + C \\
-\csc^2\theta & \xleftarrow{\frac{d}{d\theta}} & \mathbf{\cot\theta} & \xrightarrow{\int} & \ln|\sin\theta| + C \\
\frac{1}{\sqrt{1-x^2}} & \xleftarrow{\frac{d}{dx}} & \mathbf{\arcsin x} & \xrightarrow{\int} & x\arcsin x + \sqrt{1-x^2} + C \\
-\frac{1}{\sqrt{1-x^2}} & \xleftarrow{\frac{d}{dx}} & \mathbf{\arccos x} & \xrightarrow{\int} & x\arccos x - \sqrt{1-x^2} + C \\
\frac{1}{1+x^2} & \xleftarrow{\frac{d}{dx}} & \mathbf{\arctan x} & \xrightarrow{\int} & x\arctan x - \frac{1}{2}\ln(1+x^2) + C
\end{array}$$

12. Power Series Expansions

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\theta^{2n+1}$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\theta^{2n}$$

13. Limits

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$$

14. Tangent Half-Angle Substitution

$$\int f(\sin\theta, \cos\theta)\, d\theta = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2\,dt}{1+t^2} \qquad t = \tan\frac{\theta}{2}$$

15. Harmonic Addition Theorem

$$a\sin\theta + b\cos\theta = \operatorname{sgn}(b)\sqrt{a^2+b^2}\cos\left(\theta + \arctan\left(-\frac{a}{b}\right)\right)$$

16. Trigonometric Ratios of Special Angles

θ in radians	**0**	$\boldsymbol{\frac{\pi}{12}}$	$\boldsymbol{\frac{\pi}{6}}$	$\boldsymbol{\frac{\pi}{4}}$	$\boldsymbol{\frac{\pi}{3}}$	$\boldsymbol{\frac{5\pi}{12}}$	$\boldsymbol{\frac{\pi}{2}}$
in degrees	**0°**	**15°**	**30°**	**45°**	**60°**	**75°**	**90°**
sin θ	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	1
cos θ	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	0
tan θ	0	$2-\sqrt{3}$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$2+\sqrt{3}$	*undefined*

17. Reflections and Shifts

φ in radians	$\boldsymbol{\frac{\pi}{2} \pm \theta}$	$\boldsymbol{\pi \pm \theta}$	$\boldsymbol{\frac{3\pi}{2} \pm \theta}$	$\boldsymbol{2\pi \pm \theta}$
in degrees	**90° ± θ**	**180° ± θ**	**270° ± θ**	**360° ± θ**
sin φ	$\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$
cos φ	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$	$\pm\cos\theta$
tan φ	$\mp\cot\theta$	$\pm\tan\theta$	$\mp\cot\theta$	$\pm\tan\theta$
csc φ	$\sec\theta$	$\mp\csc\theta$	$-\sec\theta$	$\pm\csc\theta$
sec φ	$\mp\csc\theta$	$-\sec\theta$	$\pm\csc\theta$	$\pm\sec\theta$
cot φ	$\mp\tan\theta$	$\pm\cot\theta$	$\mp\tan\theta$	$\pm\cot\theta$

18. Composition With Inverse Trigonometric Functions

$$\sin(\arcsin x) = x \qquad \cos(\arcsin x) = \sqrt{1-x^2} \qquad \tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$$

$$\sin(\arccos x) = \sqrt{1-x^2} \qquad \cos(\arccos x) = x \qquad \tan(\arccos x) = \frac{\sqrt{1-x^2}}{x}$$

$$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}} \qquad \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}} \qquad \tan(\arctan x) = x$$

$$\sin(\operatorname{arccsc} x) = \frac{1}{x} \qquad \cos(\operatorname{arccsc} x) = \frac{\sqrt{x^2-1}}{x} \qquad \tan(\operatorname{arccsc} x) = \frac{1}{\sqrt{x^2-1}}$$

$$\sin(\operatorname{arcsec} x) = \frac{\sqrt{x^2-1}}{x} \qquad \cos(\operatorname{arcsec} x) = \frac{1}{x} \qquad \tan(\operatorname{arcsec} x) = \sqrt{x^2-1}$$

$$\sin(\operatorname{arccot} x) = \frac{1}{\sqrt{1+x^2}} \qquad \cos(\operatorname{arccot} x) = \frac{x}{\sqrt{1+x^2}} \qquad \tan(\operatorname{arccot} x) = \frac{1}{x}$$

19. De Moivre's Theorem and Complex Exponential Relations

$$e^{i\theta} = \operatorname{cis}\theta = \cos\theta + i\sin\theta$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \tan\theta = -i\,\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$$

$$\csc\theta = \frac{2i}{e^{i\theta}-e^{-i\theta}} \qquad \sec\theta = \frac{2}{e^{i\theta}+e^{-i\theta}} \qquad \cot\theta = i\,\frac{e^{i\theta}+e^{-i\theta}}{e^{i\theta}-e^{-i\theta}}$$

$$\operatorname{arccis} z = -i\ln z$$

$$\arcsin z = -i\ln\left(iz+\sqrt{1-z^2}\right) \qquad \arccos z = -i\ln\left(z+\sqrt{z^2-1}\right) \qquad \arctan z = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)$$

$$\operatorname{arccsc} z = -i\ln\left(\frac{i}{z}+\sqrt{1-\frac{1}{z^2}}\right) \qquad \operatorname{arcsec} z = -i\ln\left(\frac{1}{z}+i\sqrt{1-\frac{1}{z^2}}\right) \qquad \operatorname{arccot} z = \frac{i}{2}\ln\left(\frac{z-i}{z+i}\right)$$